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Solution to Force Problem of Linear Elasticity Theory for Quarter Space with Edge-uniform Forces

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Abstract

In the paper, the force problem of the linear elasticity theory is solved for a quarter space with edge-uniform forces. It is demonstrated that it is possible find a Green function analog that corresponds to the derivative of the delta function in the boundary conditions. In this formulation of the problem, it is reduced to the Fredholm integral equation of the second kind with a solution in the form of a rapidly convergent series. The method applied can be used to solve the force problems of the linear elasticity theory for a space sector with a dihedral angle of arbitrary value.

Keywords: Linear elasticity theory, Quarter space, Integral equation.

Introduction

Various applications are associated with problems in which surface forces are applied to regions near spatial angles of body. It is often that the applied forces have a large length along the edge of this angle and are localized close to the angle. The authors have met such a problem as an auxiliary one while calculating the condition for formation of cracks in metals under a strong thermal load. The fact is that the earlier models of crack formation in the surface layer of a material under pulsed heat load are one-dimensional [1,2]. To describe the cracks propagating along the surface [3] it is required to increase the dimensionality of the elasticity problem. In so doing, the force problem of linear elasticity theory for a quarter space with edge-uniform forces became an auxiliary problem. The transition to a two-dimensional formulation of the elasticity problem significantly increases the demands to the computing power required for modeling of elastic deformations. Therefore, the task of this paper is to reduce the system of equations to a form a numerical solutionof which it is less costly. Unfortunately, because of the non-uniform mixed boundary conditions, the attempts to use the results of the solution of the elasticity problem for a quarter of space in [4,5] were unsuccessful.

We formulate the problem mathematically and reduce the system of differential equations to the integral Fredholm equation of the second kind. The applied method is essentially a modification of the general method of boundary integral equations described in [6]. However, in view of the specific nature of the problem formulation, an integral equation can be compiled for the solution itself. Such an equation has an analytic solution in the form of a series in powers of integral operator. In this case, the series turns out to be rapidly convergent in the modulus maximum norm. While finding the solution, we will identify some problems that arise in replacement of finite body by infinite elastic medium and discuss methods to eliminate these problems.

Formulation of the problem

The equations of the linear theory of elasticity of isotropic medium without volume forces and temperature changes have the following form [7]:

$$\partial_i \partial_j u_j + (1 - 2\sigma)\partial_j \partial_j u_i = 0 \tag{1.1}$$

where u_i is the displacement and σ We will only use orthogonal coordinate systems, so we will not distinguish between covariant and contravariant indices. We solve the problem in the coordinate system depicted in Figure 1. In such a coordinate system, the medium occupies the region of $y > 0$ and $z > 0$. The boundary conditions are written as follows [7]:

$$\sigma_{zi} = -F_i^1(y) \ for \ z = 0 \ and \ y \geq 0, \tag{1.2}$$

$$\sigma_{iy} = -F_i^2(z) \ for \ y=0 \ and \ z \geq 0 \tag{1.3}$$

where σ_{ij} is the stress tensor and F_i^1 and F_i^2 are the surface forces.

The relation between the stresses and displacements is expressed by the Hooke law:

$$\sigma_{ij} = \frac{E}{1+\sigma}\left(u_{ij} + \frac{\sigma}{1-2\sigma}u_{ll}\delta_{ij}\right), \tag{1.4}$$

where E is the Young modulus and u_{ij} is the strain tensor:

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.5}$$

The equations and boundary conditions are linear with respect to the displacement. Therefore, to solve the problem with an arbitrary force $F_i^1(y)$ and a zero force $F_i^2(z)$ it would be sufficient to find the Green function [8], i.e., the solution of the problem for an external force of the following form:

$$F_i^1(y) = F_{0i}^1\delta(y-a) \tag{1.6}$$

$$F_i^2(z) = 0 \tag{1.7}$$

Because of the symmetry of the problem, the solution for an arbitrary force $F_i^2(z)$ is obtained from the desired solution by replacement of F_i^1 by F_i^2 and replacement of y by z and z by y. Hereinafter the notation F_{0i}^1 will denote the coordinate-independent amplitude of external force, each time clarifying which dependence of the force on the coordinate is currently used.

Analysis of the problem formulation

The solution of the problem (1.1), (1.6), (1.7) is supposed to be not only static (it does not involve time), but also local. This means that significant displacements occur only near the point of application of force (1.6).

If we introduce the polar coordinates R, ϕ according to Figure 2, the locality condition means that

$$\lim_{R\to\infty} u_i(R,\phi) = 0 \tag{2.1}$$

for any $\varphi \in [0;\pi/2]$. Unfortunately, there is no solution with such properties. We will show this. The equation (1.1) can be written in a divergent form:

$$\frac{\partial\sigma_{ij}}{\partial x_j} = 0. \tag{2.2}$$

It follows that the momentum flux through the closed surface for the static problem is zero. Let us calculate the momentum flux through a quarter of a cylinder with a height L and a cross section shown in Figure 2. The momentum flux through a closed surface in the static problem is equal to zero. The momentum flux through the cylinder bases is equal to zero because their normal coincides with the direction of uniformity of the problem. The rest of the flux is written as follows:

$$L\int_0^R F_i^1(y)dy + L\int_0^R F_i^2(z)dz + L\int_0^{\pi/2}\sigma_{ij}R_j d\varphi = 0. \tag{2.3}$$

The sum of the first two integrals in this expression gives the linear external force acting on part of the volume under consideration. With local forces, this value approaches a constant for sufficiently large R. The last integral can be estimated as the product of the characteristic value of the integrand by the length of the integration interval. Hence we obtain the asymptotic behavior of the stress tensor at large R:

$$\sigma ij \propto \frac{1}{R}. \tag{2.4}$$

A stress tensor in the linear theory of elasticity is expressed as a uniform first-order differential operator of displacements (expressions (1.4) and (1.5)).

Taking into account the expression (2.1), the displacements at any finite point (R,ϕ) can be obtained by integrating the value of order of the stress tensor in the appropriate limits:

$$ui(R,\varphi) - \int_R^\infty \frac{dr}{r} \tag{2.5}$$

However the integral assumes infinite value. Thus, the force (1.6) causes infinite displacements at each finite point, which is physically (and mathematically) pointless. An analogous

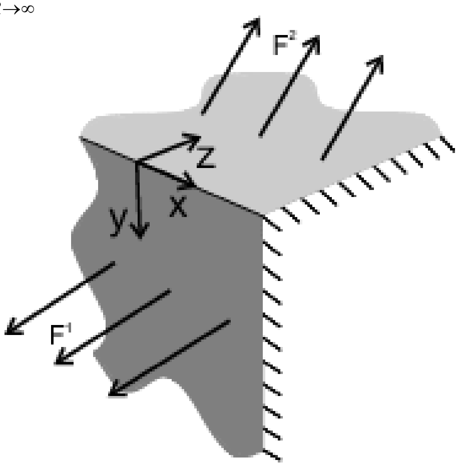

Figure 1: Quarter space filled with elastic medium and chosen coordinate system

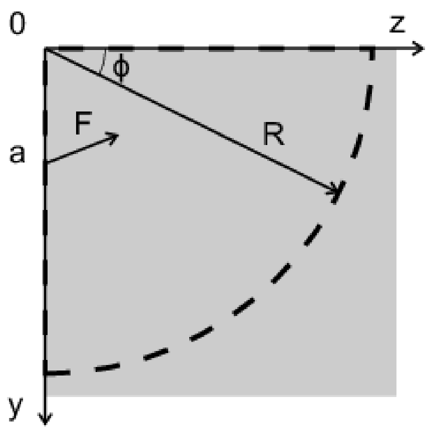

Figure 2: Cross-section of integration surface for computing asymptotic behavior of stresses

situation often arises in the plane problems of the theory of elasticity [9].

Correction of the formulation of the problem

The displacement can be finite only if the distributed external force is equal to zero. Therefore, we assume the above and solve the problem with a simplest external force the integrating of which yields zero:

$$F_i^1(y) = F_{0i}^1 \partial_y \delta(y-a) \tag{3.1}$$

$$F_i^2(z) = 0 \tag{3.2}$$

It may seem that we reduced the generality of the problem in the boundary conditions, assuming that each integral force acting on the boundaries at $y = 0$ and $z = 0$ is equal to zero. However, since these boundaries have a common point ($y = 0$, $z = 0$), to which opposing forces can be applied and referred to different parts of the boundary, it is possible to simultaneously null both linear forces acting on the half-planes. With the total distributed linear force equal to zero, the following replacement, which does not change the physical formulation of the problem, enables zeroing these two forces separately:

$$F_i^1(y) \to F_i^1(y) - \delta(y) \int_0^\infty F_i^1(a)\, da, \tag{3.3}$$

$$F_i^2(z) \to F_i^2(z) - \delta(z) \int_0^\infty F_i^2(a)\, da. \tag{3.4}$$

Let $u_i(y,z)$ and $u_{1i}(a,y,z)$ denote the solutions of equation (1.1) with boundary conditions (1.6) and (1.7) and (3.1) and (3.2), respectively. The solutions linearly depend on the amplitude of the force F_{0i}^1 in the boundary conditions, so we write this linear relationship in a general form:

$$u_{1i}(a,y,z) = u_{1i}^j(a,y,z) F_{0j}^1. \tag{3.5}$$

The introduced quantity u_{1i}^j is an analogue of the Green tensor. In what follows we will use the same form: the tensor superscript corresponds to the force amplitude index in the boundary conditions, and the tensor subscript to the displacement vector index. We will also need notation for displacements in the solution to equation (1.1) with a different geometry of the problem and other boundary conditions. We will denote them by a numerical subscript.

Now we are obtaining an expression for u_i in terms of u_{1i}^j. Due to the linearity of the problem, the function $u_{1i}^j(a,y,z)$ integrated with any kernel $h_j(a)$ with respect to a remains a solution to equation (1.1). One only needs to choose a kernel such that this transformation converts boundary conditions (3.1) into (1.2) with an arbitrary given function $F_i^1(y)$. The corresponding condition on the kernel is written as follows:

$$\int_0^\infty h_j(a) \partial_y \delta(y-a)\, da = F_j^1(y). \tag{3.6}$$

The integral is easy to take, and for $y > 0$ we have

$$\partial_y h_j(y) = F_j^1(y) \tag{3.7}$$

The desired kernel, up to a constant, is equal to

$$h_j(y) = \int_0^a F_j^1(\xi)\, d\xi. \tag{3.8}$$

The constant is chosen according to the case of zero integral force. So, the solution for an arbitrary force will be written as follows:

$$u_i(y,z) = \int_0^\infty \left(\int_0^a F_j^1(\xi)\, d\xi \right) u_{1i}^j(a,y,z)\, da \tag{3.9}$$

We have expressed the solution to the general problem $u_i(y,z)$ in terms of $u_{1i}^j(a,y,z)$, i.e., the solution of equation (1.1) with boundary conditions (3.1) and (3.2) and arbitrary direction and amplitude of the force F_{0j}^1. Next we will study the properties of $u_{1i}^j(a,y,z)$ and will find it.

With a nonzero $F_i^2(z)$ and due to the symmetry, the complete answer is written as follows:

$$u_i(y,z) = \int_0^\infty \left(\int_0^a F_j^1(\xi)\, d\xi \right) u_{1i}^j(a,y,z)\, da + \int_0^\infty \left(\int_0^a F_j^{2T}(\xi)\, d\xi \right) u_{1i}^{jT}(a,z,y)\, da, \tag{3.10}$$

where the superscript T denotes permutation of the displacement vector coordinates y and z .

Self-similarity of the solution

The problem posed in Section 4 has one characteristic dimension a . Equation (1.1) and external force (3.2) contain no characteristic dimensions. The parameter a only occurs in boundary condition (1.2) with force (3.1). To obtain from it the scaling law for the desired solution with respect to this parameter we write down boundary condition (1.2) as follows:

$$\frac{E}{2(1+\sigma)} \left(\partial_i u_{1z}(a,y,z) + \partial_z u_{1i}(a,y,z) \right) + \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \left(\partial_y u_{1y}(a,y,z) + \partial_z u_{1z}(a,y,z) \right) \delta_{zi} = -F_{0i}^1 \partial_y \delta(y-a) \tag{4.1}$$

Substituting expression (3.5), yields

$$\frac{E}{2(1+\sigma)} \left(\partial_i u_{1z}^j(a,y,z) + \partial_z u_{1i}^j(a,y,z) \right) F_{0j}^1 + \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \left(\partial_y u_{1y}^j(a,y,z) + \partial_z u_{1z}^j(a,y,z) \right) \delta_{zi} F_{0j}^1 = -\delta_i^j F_{0j}^1 \partial_y \delta(y-a) \tag{4.2}$$

Since this expression must be valid for any amplitude of the force F_{0j}^1, it can be "cancelled":

$$\frac{E}{2(1+\sigma)} \left(\partial_i u_{1z}^j(a,y,z) + \partial_z u_{1i}^j(a,y,z) \right) + \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \left(\partial_y u_{1y}^j(a,y,z) + \partial_z u_{1z}^j(a,y,z) \right) \delta_{zi} = -\delta_i^j \partial_y \delta(y-a). \tag{4.3}$$

In particular, with $a = 1$ (dimensionless variables are used):

$$\frac{E}{2(1+\sigma)}\left(\partial_i u_{1z}^j(1,y,z)+\partial_z u_{1i}^j(1,y,z)\right)+$$
$$\frac{E\sigma}{(1+\sigma)(1-2\sigma)}\left(\partial_y u_{1y}^j(1,y,z)+\partial_z u_{1z}^j(1,y,z)\right)\delta_{zi}=-\delta_i^j\partial_y\delta(y-1). \tag{4.4}$$

We transform expression (4.3):

$$\frac{E}{2(1+\sigma)}\left(\partial_i u_{1z}^j(a,y,z)+\partial_z u_{1i}^j(a,y,z)\right)+$$
$$\frac{E\sigma}{(1+\sigma)(1-2\sigma)}\left(\partial_y u_{1y}^j(a,y,z)+\partial_z u_{1z}^j(a,y,z)\right)\delta_{zi}=-\frac{1}{a}\delta_i^j\partial_y\delta\left(\frac{y}{a}-1\right). \tag{4.5}$$

From formula (4.4) we have:

$$\frac{E}{2(1+\sigma)}\left(\partial_i u_{1z}^j(a,y,z)+\partial_z u_{1i}^j(a,y,z)\right)+\frac{E\sigma}{(1+\sigma)(1-2\sigma)}$$
$$\left(\partial_y u_{1y}^j(a,y,z)+\partial_z u_{1z}^j(a,y,z)\right)\delta_{zi}=$$
$$\frac{E}{2a(1+\sigma)}\left(\partial_i u_{1z}^j\left(1,\frac{y}{a},\frac{z}{a}\right)+\partial_z u_{1i}^j\left(1,\frac{y}{a},\frac{z}{a}\right)\right)+\frac{E\sigma}{a(1+\sigma)(1-2\sigma)}$$
$$\left(\partial_y u_{1y}^j\left(1,\frac{y}{a},\frac{z}{a}\right)+\partial_z u_{1z}^j\left(1,\frac{y}{a},\frac{z}{a}\right)\right)\delta_{zi}. \tag{4.6}$$

The following scaling transformation will comply with expression (4.6):

$$u_{1i}^j(a,y,z)=\frac{1}{a}u_{1i}^j\left(1,\frac{y}{a},\frac{z}{a}\right). \tag{4.7}$$

Force along the edge

The problem has different symmetry groups for different directions of the applied force. Therefore, we will separately consider cases of different force directions using the well-known solution [7] of the linear elasticity problem for the half-space $z>0$ filled with elastic medium with the following boundary conditions:

$$F_i^1=F_{0i}^1\delta(x)\delta(y)\ \ for\ z=0 \tag{5.1}$$

Denote it by $u_{2i}(x,y)$ and introduce $u_{2i}^j(x,y)$ similarly to expression (3.5):

$$u_{2i}(y,z)=u_{2i}^j(y,z)F_{0j}^1. \tag{5.2}$$

We will write down the necessary parts of this solution in the course of the calculations. This solution complies with equation (1.1), but because of the different geometry of the problem it meets another boundary condition at $z=0$, and in this case the boundary condition at $y=0$ is not specified at all. It follows from the linearity of the problem that any linear transformation of this solution does not impair the compliance with equation (1.1). Therefore, we will look for a transformation such that the resulting surface forces at the boundaries of the quarter space are as close as possible to expressions (3.1) and (3.2).

First, we solve the problem for the quarter space with boundary conditions (3.1) and (3.2) for a force acting along the edge of the quarter space:

$$F_{0x}^1=1,\ F_{0y}^1=0,\ F_{0z}^1=0 \tag{5.3}$$

There is a known solution [7] of the problem for the half-space $z>0$ filled with elastic medium with boundary conditions (5.1) and (5.3):

$$u_{2x}^x=\frac{1+\sigma}{2\pi E}\left(\frac{2(1-\sigma)r+z}{r(r+z)}+\frac{(2r(\sigma r+z)+z^2)x^2}{r^3(r+z)^2}\right), \tag{5.4}$$

$$u_{2y}^x=\frac{1+\sigma}{2\pi E}\frac{(2r(\sigma r+z)+z^2)}{r^3(r+z)^2}xy, \tag{5.5}$$

$$u_{2z}^x=\frac{1+\sigma}{2\pi E}\left(\frac{1-2\sigma}{r(r+z)}+\frac{z}{r^3}\right)x, \tag{5.6}$$

where $r=\sqrt{x^2+y^2+z^2}$. We obtain from it a solution for the half-space with boundary conditions (3.1), (3.2) and (5.3) of interest to us (we denote it $u_{3i}^x(a,y,z)$). Let us find a linear transformation that reduces (5.1) to form (3.1). To this end, we differentiate (5.1) with respect to y, replace x by $x-x'$ and integrate with respect to x' over the real line:

$$\int_{-\infty}^{\infty}\partial_y\left(F_{0i}^1\delta(x-x')\delta(y)\right)dx'=F_{0i}^1\partial_y\delta(y) \tag{5.7}$$

To obtain (3.1) one only needs to replace y by $y-a$. From the linearity of the problem it follows that the same transformations must be performed with solutions (5.4), (5.5) and (5.6):

$$u_{3i}^x(a,y,z)=\int_{-\infty}^{\infty}\partial_y\left(u_{2i}^x(x-x',y-a)\right)dx'. \tag{5.8}$$

It is important that at first the differentiation must be carried out and then the integration, since otherwise the integration will give infinity for the reasons described in Section 3. Similar calculations show that an analogous relationship is valid for any direction of the force:

$$u_{3i}^j(a,y,z)=\int_{-\infty}^{\infty}\partial_y\left(u_{2i}^j(x-x',y-a)\right)dx'. \tag{5.9}$$

After elementary mathematical transformations we have:

$$u_{3x}^x(a,y,z)=-\frac{2(y-a)(1+\sigma)}{\pi E\left((y-a)^2+z^2\right)}, \tag{5.10}$$

$$u_{3y}^x(a,y,z)=0 \tag{5.11}$$

$$u_{3z}^x(a,y,z)=0 \tag{5.12}$$

Because of this simple expression for displacements, only σ_{xy} and σ_{xz} are non-zero components of the stress tensor. The expression $u_{3x}^x(a,y,z)$ complies with equation (1.1) and boundary condition (1.2), but σ_{xy} in boundary condition (1.3) does not vanish at $y=0$. The stress tensor component σ_{xy} vanishes if the solution has been symmetrized. Corresponding to addition of a force acting on the half-space $y<0$, this will not spoil the compliance with boundary condition (1.2). We denote the symmetrized solution u_{4i}. The corresponding expressions for the external force direction along the axis x are written as follows:

$$u_{4x}^x(a,y,z) = u_{3y}^x(a,y,z) - u_{3y}^x(-a,y,z) \tag{5.13}$$

$$u_{4y}^x(a,y,z) = 0 \tag{5.14}$$

$$u_{4z}^x(a,y,z) = 0 \tag{5.15}$$

The function $u_{4i}^x(a,y,z)$ complies with both equation (1.1) and boundary conditions (1.2) and (1.3) with expressions (3.1), (3.2) and (3.3), i.e., it is the desired $u_{1i}^x(a,y,z)$. The expression for it is written as follows:

$$u_{1x}^x(a,y,z) = \frac{4a(a^2-y^2+z^2)(1+\sigma)}{\pi E((y-a)^2+z^2)((y+a)^2+z^2)}, \tag{5.16}$$

$$u_{1y}^x(a,y,z) = 0 \tag{5.17}$$

$$u_{1z}^x(a,y,z) = 0 \tag{5.18}$$

These expressions are exact answers to the problem of the subsection. The resulting expression is some analogue of the Green function for displacements. However, the solution of the problem with an arbitrary function in the boundary condition is expressed by more complicated expression (3.9). However, for mechanical stresses, one can write a real Green function.

Let us calculate the displacements for the case of an external force:

$$F_x^1 = \delta(y-a) - \delta(y-a_1),\ F_y^1 = 0,\ F_z^1 = 0 \tag{5.19}$$

In accordance with expression (3.9), the displacement in this case is expressed by the following formula:

$$u_i(y,z) = \int_a^{a_1} u_{1i}^x(\xi,y,z)\,d\xi. \tag{5.20}$$

After calculation of integrals we have:

$$u_x(y,z) = \frac{1+\sigma}{\pi E}\ln\left((y-a_1)^2+z^2\right) + \frac{1+\sigma}{\pi E}\ln\left((y+a_1)^2+z^2\right) - \frac{1+\sigma}{\pi E}\ln\left((y-a)^2+z^2\right) - \frac{1+\sigma}{\pi E}\ln\left((y+a)^2+z^2\right), \tag{5.21}$$

$$u_y(y,z) = 0 \tag{5.22}$$

$$u_z(y,z) = 0 \tag{5.23}$$

For these displacements, only two components of the stress tensor will differ from zero: σ_{xy} and σ_{xz}. We perform the calculations for : σ_{xy}

$$\sigma_{xy}(y,z) = \frac{2}{\pi}\frac{y+a_1}{(y+a_1)^2+z^2} + \frac{2}{\pi}\frac{y-a_1}{(y-a_1)^2+z^2} - \frac{2}{\pi}\frac{y+a}{(y+a)^2+z^2} - \frac{2}{\pi}\frac{y-a}{(y-a)^2+z^2}. \tag{5.24}$$

If the point of application of the force a_1 tends to infinity, then the stress will cease depending on it:

$$\sigma_{xy}(a,y,z) = -\frac{2}{\pi}\left(\frac{y+a}{(y+a)^2+z^2} + \frac{y-a}{(y-a)^2+z^2}\right). \tag{5.25}$$

The corresponding expression for σ_{xz} is:

$$\sigma_{xz}(a,y,z) = -\frac{2z}{\pi}\left(\frac{1}{(y+a)^2+z^2} + \frac{1}{(y-a)^2+z^2}\right). \tag{5.26}$$

These expressions are the Green functions for the mechanical stress tensor. The situation in which one can write the Green function for the stresses and only some analog for the displacements is similar to calculating the potential and electric field of an infinite uniformly charged plane: the potential is infinite everywhere, while its derivative is finite everywhere.

Force normal to surface

Now we solve the problem for a force acting normally to the surface:

$$F_{0x}^1 = 0,\ F_{0y}^1 = 0,\ F_{0z}^1 = 1 \tag{6.1}$$

The solution of this problem for a half-space with force (5.1) has the following form [2]

$$u_{2x}^z = \frac{1+\sigma}{2\pi E}\left(\frac{xz}{r^3} - \frac{(1-2\sigma)x}{r(r+z)}\right), \tag{6.2}$$

$$u_{2y}^z = \frac{1+\sigma}{2\pi E}\left(\frac{yz}{r^3} - \frac{(1-2\sigma)y}{r(r+z)}\right), \tag{6.3}$$

$$u_{2z}^z = \frac{1+\sigma}{2\pi E}\left(\frac{2(1-\sigma)}{r} + \frac{z^2}{r^3}\right). \tag{6.4}$$

We can obtain $u_{3i}^z(a,y,z)$ and $u_{4i}^z(a,y,z)$ similarly to the previous section. The solution $u_{3i}^z(a,y,z)$ corresponds to boundary conditions (3.1), (3.2) and (3.1):

$$u_{3x}^z = 0 \tag{6.5}$$

$$u_{3y}^z(a,y,z) = \frac{2(1+\sigma)z((y-a)^2(\sigma-1)+z^2\sigma)}{\pi E((y-a)^2+z^2)^2}, \tag{6.6}$$

$$u_{3z}^z(a,y,z) = \frac{2(1+\sigma)(y-a)((y-a)^2(\sigma-1)+z^2(\sigma-2))}{\pi E((y-a)^2+z^2)^2}. \tag{6.7}$$

$u_{4i}^z(a,y,z)$ are y-symmetric displacements complying with equation (1.1)and boundary condition (1.2) at $z=0$:

$$u_{4x}^z = 0 \tag{6.8}$$

$$u_{4y}^z(a,y,z) = u_{3y}^z(a,y,z) - u_{3y}^z(-a,y,z) \tag{6.9}$$

$$u_{4z}^z(a,y,z) = u_{3z}^z(a,y,z) - u_{3z}^z(-a,y,z) \tag{6.10}$$

As in the previous section, the symmetrization with respect y makes it possible to nullify σ_{xy} and σ_{zy} at $y=0$, whereas σ_{yy} only doubles after this operation:

$$\sigma_{xy}(a,z) = 0, \tag{6.11}$$

$$\sigma_{yy}(a,z) = -\frac{8az(a^2-z^2)}{\pi(a^2+z^2)^3}, \tag{6.12}$$

$$\sigma_{zy}(a,z) = 0. \tag{6.13}$$

So, we have that for the displacement u_{4i}^z only the normal component of the force acting on the surface $y=0$, $z>0$ differs from the zero values set by boundary conditions (3.2). We look for a solution exactly for such external force direction. For this reason, we can express the difference of u_{4i}^z from the desired u_{1i}^z in terms of u_{1i}^z and obtain an equation for it. To this end, we use expression (3.10):

$$u_{1i}^z(a,y,z) - u_{4i}^z(a,y,z) = \int_0^\infty \left(\int_0^b \sigma_{yy}(a,\xi) d\xi \right) u_{1i}^{zT}(b,z,y) db. \tag{6.14}$$

Using expression (6.12) and taking the integral with respect to ξ, we have:

$$u_{1i}^z(a,y,z) - u_{4i}^z(a,y,z) = \int_0^\infty \left(\int_0^b \sigma_{yy}(a,\xi) d\xi \right) u_{1i}^{zT}(b,z,y) db. \tag{8.15}$$

This expression includes the $u_{1i}^z(a,y,z)$ value for different a. To simplify the expression, we can use the law of solution scaling with respect to this parameter obtained in Correction of the formulation of the problem section.

Note that the scaling law for u_{4i}^z is the same:

$$u_{4i}^z(a,y,z) = \frac{1}{a} u_{4i}^z\left(1, \frac{y}{a}, \frac{z}{a}\right). \tag{6.16}$$

Knowing the scaling law for the problem, we can look for a solution solely for $a=1$. We denote $u_{1i}(y,z) = u_{1i}(1,y,z)$ and $u_{4i}(y,z) = u_{4i}(1,y,z)$.

The x component of each term of equation (6.15) is equal to zero. With the help of (4.7), the components y and z at $a=1$ are reduced to the following form:

$$u_{1y}^z(y,z) = u_{4y}^z(y,z) - \int_0^\infty \frac{4b}{\pi(1+b^2)^2} u_{1z}^z\left(\frac{z}{b}, \frac{y}{b}\right) db, \tag{6.17}$$

$$u_{1z}^z(y,z) = u_{4z}^z(y,z) - \int_0^\infty \frac{4b}{\pi(1+b^2)^2} u_{1y}^z\left(\frac{z}{b}, \frac{y}{b}\right) db. \tag{6.18}$$

Excluding either $u_{1y}(y,z)$ or $u_{1z}(y,z)$ from equations (6.17) and (6.18), we obtain the equation for the other function. So, we have separate equations for $u_{1y}(y,z)$ and $u_{1z}(y,z)$:

$$u_{1z}^z(y,z) - \int_0^\infty\int_0^\infty \frac{16}{\pi^2} \frac{bc}{(1+b^2)^2(1+c^2)^2} u_{1z}^z\left(\frac{y}{bc}, \frac{z}{bc}\right) dbdc = u_{4z}^z(y,z) - \int_0^\infty \frac{4b}{\pi(1+b^2)^2} u_{4y}^z\left(\frac{z}{b}, \frac{y}{b}\right) db, \tag{6.19}$$

$$u_{1y}^z(y,z) - \int_0^\infty\int_0^\infty \frac{16}{\pi^2} \frac{bc}{(1+b^2)^2(1+c^2)^2} u_{1y}^z\left(\frac{y}{bc}, \frac{z}{bc}\right) dbdc = u_{4y}^z(y,z) - \int_0^\infty \frac{4b}{\pi(1+b^2)^2} u_{4z}^z\left(\frac{z}{b}, \frac{y}{b}\right) db. \tag{6.20}$$

In what follows, equations (6.19) and (6.20) will be converted synchronously and obtained from each other via replacement of the subscripts y by the subscripts z and vice versa. We will only write down one conversion. The substitution of variables $\alpha = bc$ and $\beta = b$ in the double integral results in its following form:

$$\int_0^\infty\int_0^\infty \frac{16}{\pi^2} \frac{\alpha\beta^3}{(1+\beta^2)^2(\beta^2+\alpha^2)^2} u_{1y}^z\left(\frac{y}{\alpha}, \frac{z}{\alpha}\right) d\alpha d\beta. \tag{6.21}$$

The integral with respect to β is easy to take [10]:

$$\int_0^\infty g(\alpha) u_{1y}^z\left(\frac{y}{\alpha}, \frac{z}{\alpha}\right) d\alpha, \tag{6.22}$$

with the following notation introduced:

$$g(\alpha) = \frac{16}{\pi^2} \frac{\alpha\left(1-\alpha^2+(1+\alpha^2)\ln(\alpha)\right)}{(\alpha^2-1)^3}. \tag{6.23}$$

In equations (6.19) and (6.20), the coordinates y and z only occur in the arguments of the functions in identical linear transformations. Therefore, the expressions become simpler in the polar coordinates ($y = r\sin(\phi)$, $z = r\cos(\phi)$):

$$u_{1z}^z(r,\varphi) - \int_0^\infty g(\alpha) u_{1z}^z\left(\frac{r}{\alpha}, \varphi\right) d\alpha = u_{4z}^z(r,\varphi) - \int_0^\infty \frac{4b}{\pi(1+b^2)^2} u_{4y}^z\left(\frac{r}{b}, \frac{\pi}{2} - \varphi\right) db. \tag{6.24}$$

Replacement of the integration variables $\xi = 1/\alpha$ and $a = 1/b$ transforms the equations as follows:

$$u_{1z}^z(r,\varphi) - \int_0^\infty g(\xi) u_{1z}^z(r\xi,\varphi) d\xi = u_{4z}^z(r,\varphi) - \int_0^\infty \frac{4a}{\pi(1+a^2)^2} u_{4y}^z\left(ra, \frac{\pi}{2} - \varphi\right) da. \tag{6.25}$$

We denote the resulting integral operator

$$If(r) = \int_0^\infty g(\xi) f(r\xi) d\xi. \tag{6.26}$$

Now, we introduce the modulus maximum norm:

$$\|f(r)\| = \max(|f(r)|). \tag{6.27}$$

We show that the operator I in this norm is a contraction one:

$$\|If(r)\| = \max\left|\int_0^\infty g(\xi) f(r\xi) d\xi\right| \le \int_0^\infty g(\xi) \max|f(r\xi)| d\xi = \max|f(r\xi)| \int_0^\infty g(\xi) d\xi = \int_0^\infty g(\xi) d\xi \|f(r)\| = \frac{4}{\pi^2}\|f(r)\| \approx 0.405\|f(r)\|. \tag{6.28}$$

For a continuous and bounded right-hand side, the Banach fixed-point theorem implies the existence and uniqueness of the solution to equation (6.25) [11]. Its right-hand side remains bounded for all values of the angle ϕ, which occurs in the equation as a parameter, except for $\phi = \pi/2$. In this case, the solution can be obtained from the continuity of the displacement.

The substitution $\chi = r\xi$ in the integral operator reduces equation (6.25) to the Fredholm equation of the second kind:

$$u_{1z}^z(r,\varphi) - \int_0^\infty K(\chi,r) u_{1z}^z(\chi,\varphi) d\chi = u_{4z}^z(r,\varphi) - \int_0^\infty \frac{4a}{\pi(1+a^2)^2} u_{4y}^z\left(ra, \frac{\pi}{2} - \varphi\right) da, \tag{6.29}$$

where the following notation of the kernel of the equation is introduced:

$$K(\chi,r)=\frac{g(\chi/r)}{r}. \tag{6.30}$$

The solution to such an equation is written as follows:

$$u_{1z}^{z}(r,\varphi)=(1-I)^{-1}\left(u_{4z}^{z}(r,\varphi)-\int_0^\infty\frac{4a}{\pi(1+a^2)^2}u_{4y}^{z}\left(ra,\frac{\pi}{2}-\varphi\right)da\right)=$$
$$\sum_{i=0}^{\infty}I^i\left(u_{4z}^{z}(r,\varphi)-\int_0^\infty\frac{4a}{\pi(1+a^2)^2}u_{4y}^{z}\left(ra,\frac{\pi}{2}-\varphi\right)da\right). \tag{6.31}$$

Replacement of the subscripts y by the subscripts z and vice versa in (6.31) yields the corresponding expression for $u_{1y}^{z}(r,\phi)$.

In accordance with (6.28), the terms of series (6.31) decrease at least as fast as a geometric sequence with the ratio $4/\pi^2\approx 0.405$.

Force along the axis *y*

We perform similar calculations for a force acting along the y axis:

$$F_{0x}^{1}=0, F_{0y}^{1}=1, F_{0z}^{1}=0. \tag{7.1}$$

For the half-space, the solution for this problem with force (5.1) [7]:

$$u_{2x}^{y}=\frac{1+\sigma}{2\pi E}\frac{(2r(\sigma r+z)+z^2)}{r^3(r+z)^2}xy, \tag{7.2}$$

$$u_{2y}^{y}=\frac{1+\sigma}{2\pi E}\left(\frac{2(1-\sigma)r+z}{r(r+z)}+\frac{(2r(\sigma r+z)+z^2)y^2}{r^3(r+z)^2}\right), \tag{7.3}$$

$$u_{2z}^{y}=\frac{1+\sigma}{2\pi E}\left(\frac{1-2\sigma}{r(r+z)}+\frac{z}{r^3}\right)y. \tag{7.4}$$

Similarly to the preceding section, we can obtain $u_{3i}^{y}(a,y,z)$ (the solution corresponding to boundary conditions (3.1), (3.2) and (7.1)) and $u_{4i}^{y}(a,y,z)$ (obtained from $u_{3i}^{y}(a,y,z)$ by formulas analogous to (6.8), (6.9) and (6.10)):

$$u_{3x}^{y}(a,y,z)=0 \tag{7.5}$$

$$u_{3y}^{y}(a,y,z)=\frac{2(y-a)(1+\sigma)((y-a)^2(\sigma-1)+z^2\sigma)}{\pi E((y-a)^2+z^2)^2}, \tag{7.6}$$

$$u_{3z}^{y}(a,y,z)=-\frac{2(1+\sigma)z(z^2(\sigma-1)+(y-a)^2\sigma)}{\pi E((y-a)^2+z^2)^2}, \tag{7.7}$$

$$u_{4x}^{y}=0 \tag{7.8}$$

$$u_{4y}^{y}(a,y,z)=u_{3y}^{y}(a,y,z)-u_{3y}^{y}(-a,y,z) \tag{7.9}$$

$$u_{4z}^{y}(a,y,z)=u_{3z}^{y}(a,y,z)-u_{3z}^{y}(-a,y,z) \tag{7.10}$$

Note that in this case $u_{4i}^{y}(a,y,z)$ is not symmetrized with respect to y. Because the force itself is directed along y, for symmetrization it would be necessary to use the opposite signs for the second summands. At $y=0$, the displacements $u_{4i}^{y}(a,y,z)$ at the boundary correspond to the following mechanical stresses:

$$\sigma_{xy}(a,z)=0, \tag{7.11}$$

$$\sigma_{yy}(a,z)=0, \tag{7.12}$$

$$\sigma_{zy}(a,z)=-\frac{8az(a^2-z^2)}{\pi(a^2+z^2)^3}. \tag{7.13}$$

Similarly to expressions (6.17) and (6.18), we have:

$$u_{1y}^{y}(y,z)=u_{4y}^{y}(y,z)-\int_0^\infty\frac{4b}{\pi(1+b^2)^2}u_{1z}^{y}\left(\frac{z}{b},\frac{y}{b}\right)db, \tag{7.14}$$

$$u_{1z}^{y}(y,z)=u_{4z}^{y}(y,z)-\int_0^\infty\frac{4b}{\pi(1+b^2)^2}u_{1y}^{y}\left(\frac{z}{b},\frac{y}{b}\right)db. \tag{7.15}$$

These equations only differ from (6.17) and (6.18) in the superscripts. Therefore, the final integral equations are the same:

$$u_{1z}^{y}(r,\varphi)-Iu_{1z}^{y}(r,\varphi)=u_{4z}^{y}(r,\varphi)-\int_0^\infty\frac{4a}{\pi(1+a^2)^2}u_{4y}^{y}\left(ra,\frac{\pi}{2}-\varphi\right)da. \tag{7.16}$$

The corresponding equation for $u_{1y}^{y}(r,\phi)$ is obtained from (7.16) by the replacing of the subscripts y by the subscripts z and vice versa.

An analytical answer in the form of a series is obtained analogously to the previous section.

Conclusion

The force problem of the linear elasticity theory with edge-uniform forces for a quarter space filled with elastic medium is reduced to the Fredholm equation of the second kind. For the case of a force directed along the edge, the kernel of the equation is formally equal to zero, i.e., an analytical answer is obtained in an explicit form. The existence and uniqueness of the analytical solution to these equations in the form of a rapidly convergent series is shown. It is noteworthy that in the calculations only the mirror symmetry with respect to the bisector of the spatial angle was used, not a specific value of the spatial angle of the solution region. Therefore, the method can be applied to elastic medium that fills a region bounded by two half-planes with any angle between them. For an arbitrary angle, one has to use a mirror transformation instead of replacing y by z and vice versa.

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