A Note on the Strong Unitary Uncertainty Relations

Xiaofen Huang 1, Ting-Gui Zhang 1*, Xianqing Li-Jost 2, Yuan-Hong Tao 3 and Shao-Ming Fei 2,4

1 School of Mathematics and Statistics, Hainan Normal University, Haikou, 571158, China
2 Max Planck Institute for Mathematics in the Sciences, Leipzig 04103, Germany
3 Department of Mathematics College of Sciences, Yanbian University, Yanji, 133002, China
4 School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

*Corresponding author: Ting-Gui Zhang, School of Mathematics and Statistics, Hainan Normal University, Haikou, 571158, China; E-mail: tinggui333@163.com

Abstract

Uncertainty relations satisfied by the product of variances of arbitrary n observables have attracted much attention. In a recent article [Phys. Rev.Lett.120,230402(2018)], the authors provided so called strong unitary uncertainty relations for a set of unitary matrices by using the positivity property of the Gram matrix, from which uncertainty relations satisfied by two quantum mechanical observables are obtained. We derive the explicit uncertainty relations satisfied by n quantum mechanical observables from such Gram matrix approach. By some algebraic transformations, we show that these uncertainty relations are just the same as the ones derived from a positive semi-definite Hermitian matrix generated by the mean values of n observables in [Scientific Reports 6,31192 (2016)].

PACS numbers: 03.65.Bz, 89.70.+c

Introduction

Uncertainty relations are of profound significance in quantum mechanics and also in quantum information theory such as entanglement detection [1,2], security analysis of quantum key distribution in quantum cryptography [3], nonlocality [4]. There are many ways to quantify the uncertainty of measurement outcomes. For instance, the uncertainty relations are expressed in terms of entropies [5–7], and many others to set a limit on the measurement precision.

The uncertainty relation (2) is further improved by Schrödinger [13],

\[
\langle (A)\rangle^2 \langle (AB)^2 \rangle^2 \geq \frac{1}{4} \left[ \langle AB \rangle + \langle A \rangle \langle B \rangle \right]^2 - \frac{1}{2} \left( \langle AB \rangle - \langle A \rangle \langle B \rangle \right)^2.
\]  

(3)

where \([A,B]=AB-BA\) is the commutator.

The commutator encodes the incompatibility, while the anticommutator encodes the correlations between the observables A and B. The Schrödinger uncertainty relation (3) includes both the commutator and anticommutator, and provides a better lower bound than the uncertainty relation (2).

Recently in Ref.[14], instead of quantum mechanical observables, the authors present a strong unitary uncertainty relation for n unitary operators, by using the Gram matrices. For two unitary matrices case, they prove that the strong unitary uncertainty relation implies the uncertainty relation (3) for two observables. An interesting question is what will be the uncertainty relation for arbitrary n observables. We derive the explicit uncertainty relations satisfied by n quantum mechanical observables from such Gram matrix approach used in [15].

In terms of the covariance matrices of the mean values of Hermitian operators, in [15] uncertainty relations for n observables have been also derived. It would be interesting to compare the n-observable relations from the approach used in [14] to the ones from the approach used in [15]. By a bijection map between the unitary operators and Hermitian operators, we show here that the strong unitary uncertainty relations given in [14] are equivalent to the Hermitian uncertainty relations presented in [15], although these two kinds of uncertainty relations are derived from quite different approaches. In proving this equivalence the general representations of n-observable uncertainty relations for both approaches are provided.
**Strong unitary uncertainty relations of n unitary operators**

Recently the authors in Ref. [14] presented uncertainty relations for unitary operators $U_0,U_1, U_2,\ldots, U_n$, based on the Gram matrix $G$ given by entries $G_{jk} = \langle \psi^k | \psi^j \rangle = \text{Tr} |\varphi\rangle \langle \varphi| U_j U_k |\varphi\rangle$, where $\varphi = \rho^{1/2}$, $\rho$ is a quantum state, the inner product is defined by $(A, B) = \text{Tr}(A^\dagger B)$. As $G$ is positive semi-definite, $\det G \geq 0$, one obtains the unitary uncertainty relations satisfied by the product of the variances

$$\Delta U_j^2 = 1 - |\langle U_j | U_j \rangle|^2 = |\langle U_j^\dagger U_j \rangle - |\langle U_j | U_j \rangle|.$$

In particular, for $n = 2$, one gets

$$\det A \Delta U_j^2 \geq 1 - |\langle U_j | U_j \rangle|^2 = |\langle U_j^\dagger U_j \rangle - |\langle U_j | U_j \rangle|.$$

(4)

By setting $U = e^{i\alpha}$ and $V = e^{i\beta}$ for some Hermitian matrices $A$ and $B$ and small parameter $\epsilon$, the above relation gives rise to the standard Robertson-Schrödinger uncertainty relation [12,13],

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4} | |(A, B)|^2 - (A)(B) |^2.$$

One may expect that strong uncertainty relations for general $n$ observables can be similarly derived. Indeed for three unitary operators $U$, $V$ and $W$, by direct calculation one obtains

$$\Delta U^2 \Delta V^2 \Delta W^2 \geq | |(U, V)|^2 - (U)(V)|^2 + | |(V, W)|^2 - (V)(W)|^2 + | |(U, W)|^2 - (U)(W)|^2 |$$

$$- | |(U)(V)|^2 - (U)(V)|^2 - | |(V)(W)|^2 - (V)(W)|^2 - | |(U)(W)|^2 - (U)(W)|^2 |$$

$$+ | |(U^\dagger W)|^2 - (U^\dagger W)|^2 + | |(V^\dagger W)|^2 - (V^\dagger W)|^2 + | |(U^\dagger V)|^2 - (U^\dagger V)|^2 |$$

$$- 2 \text{Re} | |(U^\dagger V)(V^\dagger W)|^2 (W^\dagger U)|^2,$$

(5)

where $\text{Re}(S)$ stands for the real part of $S$.

By setting $U = e^{i\alpha}$, $V = e^{i\beta}$ and $W = e^{i\gamma}$, we can derive the following uncertainty relation,

$$\Delta A^2 \Delta B^2 \Delta C^2 \geq | |(A, B, C)|^2 - (A)(B)(C) |^2 + | |(A)(B)|^2 - (A)(B) |^2 + | |(B)(C)|^2 - (B)(C) |^2 + | |(C)(A)|^2 - (C)(A) |^2,$$

(6)

which is just the one given in [15].

In order to compare the main results from [14] with the ones in [15], we first derive the explicit expressions of the unitary uncertainty relations for a birity operator by using the approach in [14].

With respect to the unitary matrices $U_0, U_1, U_2,\ldots, U_n$, the $(n+1)\times (n+1)$ Gram matrix $G$ is given by the entries $G_{jk} = \langle U_j^\dagger U_k \rangle$, $j,k = 0, 1, 2,\ldots, n$. Let us construct an $n \times n$ Hermitian matrix $\Sigma$ with entries given by $\Sigma_{jk} = \langle U_j^\dagger U_k \rangle - \langle U_j | U_k \rangle$, $j,k = 1, 2,\ldots, n$. Geometrically, the determinant of Gram matrix is the square of the volume of the parallelepiped formed by the vectors. In particular, the vectors are linearly independent if and only if the determinant is nonzero. It can be shown that

**Proposition 1.** $\det G = \det \Sigma$.

Proof: By using property of determinant, we have

$$\det G = \begin{vmatrix} 1 & \langle U_1 | U_1 \rangle & \ldots & \langle U_n | U_1 \rangle \\ \langle U_1^\dagger | U_1 \rangle & (U_1^\dagger U_1) & \ldots & (U_1^\dagger U_n) \\ \vdots & \vdots & \ddots & \vdots \\ \langle U_n^\dagger | U_1 \rangle & (U_n^\dagger U_1) & \ldots & (U_n^\dagger U_n) \end{vmatrix}$$

Then using Laplace expansion along the first column, we have

$$\det G = \det \Sigma.$$

In the following we denote $\text{perm}(n)$ any permutations of a list $(j_1, j_2, j_3,\ldots, j_n)$ with different elements. The sign $(j_1, j_2, j_3,\ldots, j_n)$ of a permutation $(j_1 j_2 j_3 \ldots j_n)$ is defined to be $1$ if the number of pairs of integers $(j_k j_l)$, $1 \leq j < l \leq n$, such that $j$ appears after $l$ in the list $(j_1, j_2, j_3,\ldots, j_n)$ is even, and $-1$ if the number of such pairs is odd. In other words, $\text{sign}(j_1 j_2 j_3 \ldots j_n)$ equals to $1$ ($-1$) if the natural order has been changed (even) times. For example, the sign $(12 n) = 1$.

By the definition of determinant, the determinant of the matrix $\Theta$ can be expressed as

$$\det \Theta = \sum_{(j_1, j_2, j_3,\ldots, j_n) \in \text{perm}(n)} \text{sign}(j_1, j_2, j_3,\ldots, j_n) \prod_{i=1}^{n} \langle U_{j_i}^\dagger U_{j_i} \rangle = \prod_{i=1}^{n} \langle U_{j_i}^\dagger U_{j_i} \rangle,$$

(7)

where $(j_1, j_2, j_3,\ldots, j_n)$ denotes the permutation of $(1, 2,\ldots, n)$, $j_1, j_2, j_3,\ldots, j_n = 1, 2,\ldots, n$. The matrix $\Theta$ is the covariance matrix of vectors $U_1, U_2,\ldots, U_n$ and is positive semi-definite. Noting that $\det \Theta = 0$ if and only if $\Theta$ is a Hermitian operator.

**Theorem 1.** $\Delta U_1^2 \Delta U_2^2 \Delta U_3^2 \Delta U_4^2 \geq \langle U_1^\dagger U_2 \rangle \langle U_2^\dagger U_3 \rangle \langle U_3^\dagger U_4 \rangle \langle U_4^\dagger U_1 \rangle$.

(8)

Remark. The inequalities (4) and (5) can be directly derived from (8) by substituting $\Sigma_{jk} = \langle U_j^\dagger U_k \rangle - \langle U_j | U_k \rangle$ into (8).

**Uncertainty relations of n observables from covariance-matrix approach**

In quantum mechanics one measures quantum mechanical observables which are Hermitian operators. In Ref. [15] the authors considered the positive semi-definite Hermitian matrix $\Theta$ generated by $n$ observables $A_1, A_2,\ldots, A_n$, with entries given by $m_{jk} = \langle A_j^\dagger A_k \rangle = \langle A_j A_k \rangle = \langle A_k A_j \rangle$, $i,s = 1, 2,\ldots, n$. To compare the relation (8) with the one in [15], we rewrite the $n$-observable uncertainty relation in [15].

$$\det \Theta = \sum_{(j_1, j_2, j_3,\ldots, j_n) \in \text{perm}(n)} \text{sign}(j_1, j_2, j_3,\ldots, j_n) \prod_{i=1}^{n} m_{j_i, j_i} = \prod_{i=1}^{n} m_{j_i, j_i},$$

(9)

Since the diagonal entries of $\Theta$ are just the variances of $A_i$ defined by $\Delta A_i^2 = \langle A_i^2 \rangle - \langle A_i \rangle^2$, from the fact that $\Theta$ is positive semi-definite, one gets from (9) the following uncertainty relations,

$$\Delta A_1^2 \Delta A_2^2 \Delta A_3^2 \Delta A_4^2 \geq \sum_{(j_1, j_2, j_3,\ldots, j_n) \in \text{perm}(n)} \text{sign}(j_1, j_2, j_3,\ldots, j_n) \prod_{i=1}^{n} m_{j_i, j_i}.$$

(10)

**Relations between (8) and (10)**

Next we show that the relations (8) in Theorem 1 from [14] and the relations (10) in Theorem 2 from [15] are equivalent.

Concerning the one to one map between the unitary operators and the Hermitian operators, for an unitary operator $U_j$, there is a Hermitian operator $A_j$, satisfying $U_j = e^{i\alpha_j}$, $j = 1, 2,\ldots, n$. Taking Taylor expansions of $U_j = e^{i\alpha_j}$, $j = 1, 2,\ldots, n$.

$$U_j = 1 + i\alpha_j + \frac{1}{2!}i^2 \alpha_j^2 + \alpha_j^3,$$

(11)

we have

\[ \text{Citation:} \text{ Xiaofan Huang, Ting-Gui Zhang, Xianqing Li-Jost, Yuan-Hong Tao and Shao-Ming Fei (2019) A Note on the Strong Unitary Uncertainty Relations. J Apl Theol 3(1): 1-3. doi: https://doi.org/10.24218/jatpr.2019.18.} \]
\[ \Delta U_{ij}^2 = e^{2\epsilon} \Delta i_{ij}^2 + O(\epsilon^3), \]

\[ g_n = e^{2\epsilon} m_n + O(\epsilon^3). \]  

Combining (8) and (12), we get

\[ e^{2\epsilon} \Delta i_{ij}^2 \Delta i_{kl}^2 + O(e^{2\epsilon}) \geq e^{2\epsilon} \sum_{j=1,j \neq i}^{n} \text{sign}(j/l,m_{ij} - m_{ik}) + O(e^{2\epsilon}). \]

(13)

(dividing by \( e^{2\epsilon} \) and taking the limit \( \epsilon \to 0 \), which gives rise to the inequality (10).

Conversely, from (10) and taking into account that \( \Delta A_{ij} = e^{2\epsilon} \Delta i_{ij}^2 + O(\epsilon) \) and \( m_n = e^{2\epsilon} m_n + O(\epsilon) \) one gets the inequality (8).

Therefore, according to the bijection map \( U = e^{2\epsilon} \) between unitary operators and Hermitian operators, the unitary uncertainty relation (8) and the Hermitian uncertainty relation (10) are equivalent.

**Conclusion**

In Ref. [14], the authors presented the unitary uncertainty relations based on the Gram matrices of multi unitary operators. In [15] the Hermitian uncertainty relations are given based on the covariance matrices of multi Hermitian operators. By deriving the explicit uncertainty relations satisfied by \( n \) quantum mechanical observables from such Gram matrix approach, we have shown that the two kinds of uncertainty relations are equivalent. The related derivations and proofs may highlight further investigations on the uncertainty relations satisfied by the product of variances of \( n \) quantum mechanical observables.

**Acknowledgments**

We are grateful to Bing Yu and Ya Xi for fruitful discussions. This work is supported by the NSF of China under Grant Nos. 11861031, 11761073 and 11675113, and Beijing Municipal Commission of Education (KZ201810028042).

**References**


