VERIZONA PUBLISHER
Publishing Your Ideas

www.verizonaonlinepublishing.com

Vol: 3, Issue: 2

Journal of Applied and Theoretical Physics Research

Relativistic Studies of Dirac Equation with a Spin-orbit Coupled Hulthen Potential including a Coulomb-like Tensor Interaction

Akaninyene D. Antia*, **Christian C. Eze, Ituen B. Okon and Louis E. Akpabio**

Theoretical Physics Group, Department of Physics, University of Uyo, Nigeria

*Corresponding author: Akaninyene D. Antia, Theoretical Physics Group, Department of Physics, University of Uyo, Nigeria; E mail: antiacauchy@yahoo.com; akaninyeneantia@uniuyo.edu.ng

Article Type: Research, Submission Date: 14 September 2018, Accepted Date: 16 October 2018, Published Date: 16 September 2019.

Citation: Akaninyene D. Antia, Christian C. Eze, Ituen B. Okon and Louis E. Akpabio (2019) Relativistic Studies of Dirac Equation with a Spin-orbit Coupled Hulthen Potential including a Coulomb-like Tensor Interaction. J Apl Theol 3(2): 1-8. doi: https://doi.org/10.24218/jatpr.2019.19.

Abstract

The relativistic symmetries of Dirac equation with a spin-orbit coupled with Hulthen potential under Coulomb-like tensor interaction have studied. The energy eigenvalues and the corresponding eigenfunctions of this system for the spin and pseudospin symmetries have also been obtained using the parametric generalization of Nikiforov-Uvarov method. Also, in this work, the numerical bound state energy for pseudospin symmetry with fixed pseudo coupling constant has been computed.

Keywords: Dirac equation, Spin-orbit, Hulthen potential, Spin symmetries, Tensor interaction.

PACS Nos: 03.65Ge, 03.65Pm, 03.65Db.

Introduction

The Dirac equation can be solved in both spin and pseudo-spin (p-spin) symmetric limits. In nuclear and hadronic spectroscopy. The spin and P-spin symmetries of the Dirac Hamiltonian can be recognized empirically [1]. The spin symmetry bears significance in studying meson-meson interactions [2] whereas the p-spin symmetry is relevant in studying deformed and super deformed nuclei in the shell-model of nuclear models [3-5]. The spin symmetry occurs when the differences between the scalar $S(r)$ and vector $V(r)$ potentials are constant, i.e $\Delta(r)=C_s$ while the P-spin symmetry occurs when the sum of the scalar and vector potentials are constant, i.e $\sum(r)=C_{ps}$ [6-7]. The P-spin symmetry refers to a quasi-degeneracy of single nuclear doublets with non-relativistic quantum numbers $\left(n,l,j=l+\frac{1}{2}\right)$ and $\left(n-1,l+2,j=l+\frac{3}{2}\right)$, where n,l and j are the single nucleon radial, orbital and total angular momentum quantum numbers respectively [8-10]. Also, in P-spin symmetry, the total angular momentum $j=\tilde{l}+\tilde{s}$ where $\tilde{l}=l+1$ is the pseudo-angular momentum and $\tilde{j}$ is the P-spin angular momentum [11]. The exact P-spin symmetry is discussed in the relativistic harmonic oscillator potential and in Woods-Saxon potentials by Chen et al., [14]. An overview of the P-spin symmetry can be found in Ref. [15-24]. Lisbon et al., [12] have studied a generalized relativistic harmonic oscillator for spin-½ particles by considering a Dirac Hamiltonian that contains quadratic spin and P-spin symmetric limits.

However, the tensor interactions where introduced into the Dirac Hamiltonian by the substitution $\hat{p}\rightarrow\hat{p}-im\omega\beta\hat{r}U(r)$ and a spin-orbit coupling (SOC) is added to the Dirac Hamiltonian [13,25,26]. Aydogdu and Sever [27] obtained exact solutions of the Dirac equation for the pseudo-harmonic potential in the presence of a linear tensor interaction under the P-spin symmetry and showed that tensor interactions remove all degeneracies between numbers of P-spin doublets. Over the past few years, the Schrödinger, Klein-Gordon (KG) and the Dirac equations have been solved for various types of potential models by difference authors [28-49] with different methods [50-54].

The Hulthen potential, being a short-ranged potential, [55-57] is a special case of the multi-parameter exponential-type potential model [58,59]. It is of the form [60]

$$V_H(r)=\frac{-V_0}{e^{\alpha r}-1},\ V_0=Ze^2\alpha \tag{1}$$

where V_0 is the amplitude of the potential, α is the screening parameter, which characterizes the range of the potential. Accordingly, the SOC interaction is defined as

$$V_{LS}(r)=V_{LS}^{(o)}\left(\frac{r_0}{\hbar}\right)^2,\frac{1}{r}\left(\frac{dV_H}{dr}\right)\left(\hat{i}.\hat{j}\right)=\frac{\alpha\delta V_0}{2r\left(e^{\alpha r}-1\right)^2}\left\langle\left(\hat{j}^2-\hat{i}^2-\hat{s}^2\right)\right\rangle \tag{2}$$

or

$$V_{LS}(r)=\frac{\alpha\delta V_0}{2r\left(e^{\alpha r}-1\right)^2}\left(j\left(j+1\right)-l\left(l+1\right)-\frac{3}{4}\right)$$

where $\delta=V_{LS}^{(o)}\left(\frac{r_0}{\hbar}\right)^2$, $V_{LS}^{(o)}=0.44V$, $V=40.5+0.14A$ and $r_0=1.2\,fm$ is the radius of the spherical nucleus in the mean-field shell model [46]. At this point, it becomes necessary to reduce the adjustable parameters r_0' by using the approximation $\frac{1}{r}\approx\frac{e^{\alpha r}}{r_0'}$ to the SOC interaction. Thus, we write the Hulthén potential with SOC as

$$V(r)=\frac{-V_0e^{-\alpha r}}{1-e^{-\alpha r}}+\frac{\alpha\delta V_0e^{-\alpha r}}{2r_0\left(1-e^{-\alpha r}\right)^2}\left(j(j+1)-l(l+1)-\frac{3}{4}\right)$$

$$=\frac{-V_0e^{-\alpha r}}{1-e^{-\alpha r}}-\frac{\alpha\delta V_0e^{-\alpha r}}{2r_0'\left(1-e^{-\alpha r}\right)^2}\begin{cases}l+1, for\ j=l-\frac{1}{2}\\ l\ ,\quad for\ j=l+\frac{1}{2}\end{cases}\tag{4}$$

The Nikiforov-Uvarov (NU) method

The Nikiforov-Uvarov (NU) method was presented by Nikiforov and Uvarov [50] and parameterized by [61] to solve linear second-order generalized parametric differential equations, usually of hypergeometric-type of the form

$$\psi''(s)+\frac{c_1-c_2s}{s\left(1-c_3s\right)}\psi'(s)+\frac{1}{s^2\left(1-c_3s\right)^2}\left[-\xi_1s^2+\xi_2s-\xi_3\right]\psi(s)=0\tag{5}$$

With the following eigen solution

$$\psi(s)=N_ns^{c_{12}}\left(1-c_3s\right)^{-c_{12}-\left(\frac{c_{13}}{c_3}\right)}P_n^{\left(c_{10}-1,\left(\frac{c_{11}}{c_3}\right)-c_{10}-1\right)}\left(1-2c_3s\right)\tag{6}$$

and the eigenvalue condition

$$c_2n-(2n+1)c_5+\left(2n+1\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+n\left(n-1\right)\right)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0\tag{7}$$

where

$$c_4=\frac{1}{2}(1-c_1),\ c_5=\frac{1}{2}(c_2-2c_3),\ c_6=c_5^2+\xi_1,\ c_7=2c_4c_5-\xi_2,\ c_8=c_4^2+\xi_3,\ c_9=c_3c_7+c_3^2c_8+c_6,$$
$$c_{10}=c_1+2c_4+2\sqrt{c_8},c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right),\ c_{12}=c_4+\sqrt{c_8},\ c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)\tag{8}$$

Solution of the Dirac equation

The Dirac equation for fermionic massive spin-½ particles moving in the field of an attractive scalar $S(r)$, a repulsive vector $V(r)$ and a tensor $U(r)$ potentials (in natural units $\hbar=c=1$) is given by the following matrix equation

$$\begin{pmatrix}\left(M+\delta(r)\right)\xi(\vec{r})-i\hat{\sigma}.\left(\nabla+\hat{r}U(r)\right)\phi(\vec{r})\\ -i\hat{\sigma}.\left(\nabla+\hat{r}U(r)\right)\phi(\vec{r})-\left(M+\delta(r)\right)\xi(\vec{r})\end{pmatrix}=\begin{pmatrix}\left(E-V(r)\right)\xi(\vec{r})\\ \left(E-V(r)\right)\xi(\vec{r})\end{pmatrix}\tag{9}$$

As a matter of convenience, we introduce the spin-orbit operator $\hat{\kappa}=\left(\hat{\sigma}.\hat{L}+1\right)$ where $\hat{L}$ is the orbital angular momentum operator. The eigenvalues of the spin-orbit operator are $\kappa=\left(j+\frac{1}{2}\right)>0$ and $\kappa=\left(j+\frac{1}{2}\right)<0$ for un aligned spin $j=l-\frac{1}{2}$ and aligned spin $j=l+\frac{1}{2}$ respectively. The quasi degenerate doublet structure can be expressed in terms of a P-spin angular momentum $\tilde{s}=\frac{1}{2}$ and a pseudo-orbital angular momentum $\tilde{l}$ which is defined as

$$\kappa=\begin{cases}-\tilde{l}=-\left(j+\frac{1}{2}\right), & for\ j=\tilde{l}-\frac{1}{2};\ aligned\ p-spin\ (\kappa<0)\\ +\left(\tilde{l}+1\right)=-\left(j+\frac{1}{2}\right), & for\ j=\tilde{l}+\frac{1}{2};\ unaligned\ p-spin\ (\kappa>0)\end{cases}$$

where $\kappa=\pm1,\pm2,\ldots$

In the Pauli-Dirac representation, the spinors are written as

$$\psi(\vec{r})=\begin{pmatrix}\xi(\vec{r})\\ \phi(\vec{r})\end{pmatrix}=\frac{1}{r}\begin{pmatrix}F_{n\kappa}(r)Y_{jm}^{l}(\theta,\varphi)\\ iG_{n\kappa}(r)Y_{jm}^{\tilde{l}}(\theta,\varphi)\end{pmatrix}\tag{10}$$

where $F_{n\kappa}$ and $G_{n\kappa}$ are the upper and lower radial wave functions respectively and $Y_{jm}^{l}(\theta,\varphi)$ and $Y_{jm}^{\tilde{l}}(\theta,\varphi)$ are spin and p-spin spherical harmonic respectively, and m is the projection of the angular momentum on the z-axis. The spin spherical harmonic are related to the spherical harmonic by [62].

$$Y_{jm}^{l}(\theta,\varphi)=\sum_{m,m_s}\left(l\ \frac{1}{2}\ j|m-m_s\ m_s\ m\right)Y_{lm-m_s}(\theta,\varphi)\chi_{\frac{1}{2}m_s}\tag{11}$$

where $m_s=\pm\frac{1}{2}$ is the spin quantum number, and the Clebsch-Gordon (CG) coefficient $\left(l\ \frac{1}{2}\ j|m-m_s\ m_s\ m\right)$ are the coupling factors and the spin states $\chi_{\frac{1}{2}m_s}$ are

$$\chi_{\frac{1}{2},\frac{1}{2}}=\begin{pmatrix}1\\0\end{pmatrix},\ \chi_{\frac{1}{2},-\frac{1}{2}}=\begin{pmatrix}0\\1\end{pmatrix}$$

Since the spin (½) and the orbital angular momentum (l) are coupled to the total angular momentum j, we give the explicit form of the $Y_{jm}^{l}(\theta,\varphi)$ for the useful cases $j=l+\frac{1}{2}$ and $j=l-\frac{1}{2}\left(j\geq\frac{1}{2}\right)$

$$Y_{l+\frac{1}{2},m}^{l}(\theta,\varphi)=\begin{pmatrix}\sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}Y_{l,m-\frac{1}{2}}(\theta,\varphi)\\ \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}Y_{l,m+\frac{1}{2}}(\theta,\varphi)\end{pmatrix}\tag{12}$$

$$Y_{l-\frac{1}{2},m}^{l}(\theta,\varphi)=\begin{pmatrix}-\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}Y_{l,m-\frac{1}{2}}(\theta,\varphi)\\ \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}Y_{l,m+\frac{1}{2}}(\theta,\varphi)\end{pmatrix}\tag{13}$$

where the above root functions result from an explicit computation of the (CG) coefficients. The p-spin spherical harmonics are likewise given by

$$Y_{jm}^{\tilde{l}}(\theta,\varphi)=\sum_{m=\pm\frac{1}{2}}\left(\tilde{l}\ \frac{1}{2}\ j|m-m_s\ m_s\ m\right)Y_{\tilde{l}m-m_s}(\theta,\varphi)\chi_{\frac{1}{2}m_s}\tag{14}$$

where $\tilde{l}=2j-l$ is the pseudo-orbital angular momentum quantum number, so that $Y_{\tilde{l}+\frac{1}{2},m}^{\tilde{l}}(\theta,\varphi)$ and $Y_{\tilde{l}-\frac{1}{2},m}^{\tilde{l}}(\theta,\varphi)$ become the unaligned p-spin spherical harmonics, respectively.

By substituting Eq. (10) into (9) and making use of the following relation $\left(\hat{\sigma}.\vec{A}\right)\left(\hat{\sigma}.\vec{B}\right)=\vec{A}.\vec{B}+i\hat{\sigma}.\left(\vec{A}\times\vec{B}\right)$ for arbitrary vectors $\vec{A}$ and $\vec{B}$, and

$$\left.\begin{aligned}&\left(\hat{\kappa}-1\right)Y_{jm}^{\tilde{l}}(\theta,\varphi)=\left(\kappa-1\right)Y_{jm}^{\tilde{l}}(\theta,\varphi)\\ &\left(\hat{\kappa}-1\right)Y_{jm}^{\tilde{l}}(\theta,\varphi)=-\left(\kappa-1\right)Y_{jm}^{i}(\theta,\varphi)\\ &\left(\hat{\sigma}.\hat{r}\right)Y_{jm}^{\tilde{l}}(\theta,\varphi)=-Y_{jm}^{i}(\theta,\varphi)\\ &\left(\hat{\sigma}.\hat{r}\right)Y_{jm}^{i}(\theta,\varphi)=-Y_{jm}^{\tilde{l}}(\theta,\varphi)\\ &\hat{\sigma}.\hat{p}=\hat{\sigma}.\hat{r}\left(\hat{r}.\hat{p}+\frac{i\hat{\sigma}.\hat{l}}{r}\right)\end{aligned}\right\}\tag{15}$$

One obtains the two coupled differential equations whose solutions are the lower $G_{n\kappa}(r)$ and upper $F_{n\kappa}(r)$ radial wave functions.

$$\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)=\left(M+E_{n\kappa}-\Delta(r)\right)G_{n\kappa}(r) \quad (16)$$

$$\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)G_{n\kappa}(r)=\left(M+E_{n\kappa}-\Delta(r)\right)F_{n\kappa}(r) \quad (17)$$

where

$$\Delta(r)=V(r)-S(r) \quad (18)$$

$$\sum(r)=V(r)+S(r) \quad (19)$$

By eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ from Eqs. (16) and (17), we obtain the two Schrödinger-like differential equations for the upper and lower radial spinor components

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa U(r)}{r}-\frac{dU(r)}{dr}-U^2(r)\right\}F_{n\kappa}(r)+\frac{\frac{d\Delta(r)}{dr}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)$$
$$=\left\{\left(M+E_{n\kappa}-\Delta(r)\right)\left(M+E_{n\kappa}-\sum(r)\right)\right\}F_{n\kappa}(r) \quad (20)$$

$$\left\{\frac{d}{dr^2}-\frac{\kappa(\kappa\quad 1)}{r^2}+\frac{2\kappa U(r)}{r}-\frac{dU(r)}{dr}-U\quad(r)\right\}G_{n\kappa}(r)+\frac{\frac{d\sum(r)}{dr}}{M+E\quad-\sum(r)}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)G_{n\kappa}(r)$$
$$=\left\{\left(M+E_{n\kappa}-\Delta(r)\right)\left(M-E_{n\kappa}+\sum(r)\right)\right\}G_{n\kappa}(r) \quad (21)$$

Spin symmetric limit

In the spin symmetric limit, $\frac{d\quad(r)}{dr}=0 \; or \; \Delta(r)=c =$ constant [6], with $\sum(r)$ taken as our potential (4) and $U(r)$ is the Coulomb-like tensor interaction, that is

$$\sum(r)=-\frac{V_0e^{-\alpha r}}{1-e^{-\alpha r}}+\frac{\alpha\delta V_0e^{-\alpha r}}{2r_0^1\left(1-e^{-\alpha r}\right)^2}\left(j(j+1)-l(l+1)-\frac{3}{4}\right) \quad (22)$$

$$U(r)=-\frac{H}{r},\; H=\frac{z_az_be^2}{4\pi\varepsilon_0},\; r\geq R_c \quad (23)$$

where $R_c=7.78\,fm$ is the Coulomb radius, z_a and z_b denote the charges of the projectile a and target nuclei b, respectively [11,13]. Thus, Eq. (20) takes the form:

$$\left\{\frac{d^2}{dr^2}-\frac{l_\kappa(l_\kappa-1)}{r^2}-\beta_{n\kappa}\left[\frac{-V_0e^{-\alpha r}}{1-e^{-\alpha r}}+\frac{\alpha\delta V_0e^{-\alpha r}}{2r_0'\left(1-e^{-\alpha r}\right)^2}\left(j(j+1)-l(l+1)-\frac{3}{4}\right]-Y_{n\kappa}^2\right\}F_{n\kappa}(r)=0 \quad (24)$$

where

$$l_\kappa=\kappa+H,\; \beta_{n\kappa}=M+E_{n\kappa}-c_s,\; \gamma_{n\kappa}^2=\left(M+E_{n\kappa}-c_s\right)\left(M-E_{n\kappa}\right)$$

To deal with the centrifugal barrier, we use a new improved approximation scheme of the form [63]

$$\frac{1}{r^2}\approx\alpha^2\left[c_0+\frac{e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)^2}\right] \quad (25)$$

Thus Eq. (24) takes on the more analytic form

$$\left\{\frac{d^2}{dr^2}-\alpha^2 l_\kappa(l_\kappa-1)\left(c_0\frac{e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)^2}\right)-\beta_{n\kappa}\left[\frac{-V_0e^{-\alpha r}}{1-e^{-\alpha r}}+\frac{\alpha\delta V_0e^{-\alpha r}}{2r_0'\left(1-e^{-\alpha r}\right)^2}\left(j(j+1)-l(l+1)-\frac{3}{4}\right]-\gamma_{n\kappa}^2\right\}F_{n\kappa}(r)=0 \quad (26)$$

By using the transformation $e^{-\alpha r}\to s,$ Eq. (26) is transformed into the following hypergeometric type equation:

$$F_{n\kappa}''(s)+\frac{(1-s)}{s(1-s)}F_{n\kappa}'(s)+\frac{1}{s^2(1-s)^2}\left[-\left(\eta_\kappa+v+\sigma^2\right)s^2+\left(2\eta_\kappa+\Lambda_\kappa+v+2\sigma^2-\chi_{n\kappa}\right)s-\left(\eta_\kappa+\sigma^2\right)\right]F_{n\kappa}(s)=0$$

With the following identifications

$$\eta_\kappa=l_\kappa(l_\kappa+1)c_0,\; v=\frac{\beta_{n\kappa}V_0}{\alpha^2},\; \sigma=\frac{\gamma_{n\kappa}}{\alpha},\; \Lambda_\kappa=l_\kappa(l_\kappa+1)$$

and

$$\chi_{n\kappa}=\begin{cases}-\dfrac{r_0^2V_{LS}^{(o)}Ze^2\beta_{n\kappa}(l+1)}{2r_0'\hbar^2}, & for\;\kappa=\left(j+\dfrac{1}{2}\right)>0\\[2ex] -\dfrac{r_0^2V_{LS}^{(o)}Ze^2\beta_{n\kappa}l}{2r_0'\hbar^2}, & for\;\kappa=-\left(j+\dfrac{1}{2}\right)<0\end{cases}$$

Comparing Eq. (27) with Eq. (5), we obtain the following parameters

$$\xi_1=\eta_\kappa+v+\sigma^2,\; \xi_2=2\eta_\kappa-\Lambda_\kappa+v+2\sigma^2-\chi_{n\kappa},\; \xi_3=\eta_\kappa+\sigma^2,\; c_1=c_2=c_3=c=1, c_4=0, c_5=-\frac{1}{2},$$
$$c_6=\frac{1}{4}+\eta_\kappa+v+\sigma^2, c_7=\Lambda_\kappa-2\eta_\kappa-v-2\sigma^2+\chi_{n\kappa}, c_8=\eta_\kappa+\sigma^2, c_9=\frac{1}{4}+\Lambda_\kappa+\chi_{n\kappa},\; c_{10}=1+2\sqrt{\eta_\kappa+\sigma^2}$$
$$c_{11}=2+2\left(\sqrt{\frac{1}{4}+\Lambda_\kappa+\chi_{n\kappa}}+\sqrt{\eta_\kappa+\sigma^2}\right), c_{12}=\sqrt{\eta_\kappa+\sigma^2}, c_{13}=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}+\Lambda_\kappa+\chi_{n\kappa}}+\sqrt{\eta_\kappa+\sigma^2}\right) \quad (28)$$

Thus, energy eigenvalues equation for the Dirac particle states is obtained as

$$E_{n\kappa}=\frac{\alpha^2}{\beta_{n\kappa}}\left\{l_\kappa(l_\kappa+1)c_0-\left(\frac{\frac{Z\beta_{n\kappa}}{2\alpha}}{\left(n+\frac{1}{2}+\delta_\kappa^\pm\right)}-\frac{\left(n+\frac{1}{2}+\delta_\kappa^\pm\right)}{2}\right)^2\right\}+M, \quad (29)$$

With $\delta_\kappa^\pm=\sqrt{\left(l_\kappa+\frac{1}{2}\right)^2+\chi_{n\kappa}^\pm}$

The reality of the eigenvalues can be implied by imposing the condition $\left(l_\kappa+\frac{1}{2}\right)^2\geq\chi_{n\kappa}^\pm$ so that the allowed values for the adjustable parameter $r_0'\geq\frac{\left(\kappa+H+\frac{1}{2}\right)^2 r_0^2V_{LS}^{(o)}ze^2\beta_{n\kappa}l_{\max}}{2\hbar^2}$ for both $\kappa<0$ and $\kappa>0$. Here, $l_{\max}$ is the maximum value of $l(i.e,\, n-1)$.

By obtaining the parametric combinations in (6), we obtain the upper radial wave function as

$$F_{n\kappa}(r)=N_n\left(e^{-\alpha r}\right)^a\left(1-e^{-\alpha r}\right)^{b+\frac{1}{2}}P_n^{(2a,2b)}\left(1-2e^{-\alpha r}\right) \quad (30)$$

with

$$a=\sqrt{\eta_\kappa+\sigma^2}$$
$$b=\sqrt{\frac{1}{4}+\Lambda_\kappa+\chi_{n\kappa}}$$

Thus, the upper Dirac spinor can be expressed as

$$\xi(\vec{r})=N_n'r^{-1}\left(e^{-\alpha r}\right)^a\left(1-e^{-\alpha r}\right)^{b+\frac{1}{2}}{}_2F_1\left(-n,n+2a+2b+1,2a+1;\,e^{-\alpha r}\right)Y_{jm}^l(\theta,\varphi), \quad (31)$$

where

$$N_n'=\frac{\Gamma(n+2a+1)}{n!\Gamma(2a+1)}N_n$$

is a new normalization constant

and the hypergeometric function [50, 64]

$${}_pF_q\left(\alpha_1,\alpha_2,\ldots,\alpha_p,\beta_1,\beta_2,\ldots,\beta_q;z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_1)_n z^n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_1)_n\, n!}$$

are related to the Jacobi polynomials $P^{(\alpha,\beta)}(.)$ through [64, 65]

$$P_n^{(\alpha,\beta)}(z)=\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}\,{}_2F_1\left(-n,\,n+\alpha+\beta+1,\,\alpha+1;\,\frac{1-z}{2}\right)$$

The normalization constant is obtained from the normalization condition

$$\int\left|\xi(\vec{r})\right|^2 r^2 dr d\Omega = -\frac{-N'_n, N_n}{\alpha}\int ds d\Omega\, s^{a-1}\left(s'\right)^a\left(1-s\right)^{b+\frac{1}{2}}\left(1-s'\right)^{b+\frac{1}{2}}\,{}_2F_1\left(-n', n'+2a+2b+1, 2a+1; s'\right)$$
$$\times\,{}_2F_1\left(-n, n+2a+2b+1, 2a+1; s\right)Y^{l*}_{j'm'}\left(\theta,\varphi\right)Y^{l}_{jm}\left(\theta,\varphi\right) = \delta\left(s'-s\right)\delta_{n'n}\delta_{l'l}\delta_{m'm}\delta_{j'j} \tag{33}$$

where $s' = e^{-\alpha r'}$ for $0 \le s' \le 1$ is understood, with $0 \le \theta' \le \pi, 0 \le \varphi' \le 2\pi, d\Omega = \sin\theta d\theta d\varphi$ and δ is the Dirac delta function. By using the following orthogonality property of the CG coefficients

$$\sum_{m,m_s}\left(i\ \frac{1}{2}\ j\ \middle| m-m_s m_s m\right)\left(i\ \frac{1}{2}\ j'\middle| m'-m_s m_s m'\right)^* = \delta_{jj'}\delta_{mm'}$$

together with the integral [66,67]

$$\int_0^1\left(1-z\right)^{2(\delta+1)}z^{2\lambda-1}\,{}_2F_1\left(-n, n+2(\delta+\lambda+1), 1+2\lambda; z\right)^2 ds$$
$$= \frac{\left(n+\delta+1\right)n!\,\Gamma\left(n+2\delta+2\right)\Gamma\left(2\lambda\right)\Gamma\left(2\lambda+1\right)}{\left(n+\delta+\lambda+1\right)\Gamma\left(n+2\lambda+1\right)\Gamma\left(2(\delta+\lambda+1)+n\right)}$$

The normalization constant is found to be

$$N'_n = \frac{\Gamma\left(n+2a+1\right)}{n!\,\Gamma\left(2a+1\right)}A_n^{a,b} \tag{34}$$

where

$$A_n^{a,b} = \sqrt{\frac{\left(n+a+b+\frac{1}{2}\right)\Gamma\left(n+2\alpha+1\right)\Gamma\left(n+2a+2b+1\right)}{n!\left(n+b+\frac{1}{2}\right)\Gamma\left(n+2b+1\right)\Gamma\left(2a\right)\Gamma\left(2a+1\right)}} \tag{35}$$

To obtain the lower spinor, we rewrite Eq. (16) as

$$G_{n\kappa}(r) = \frac{1}{M+E_{n\kappa}-c_s}\left(\frac{d}{dr}+\frac{l_\kappa}{r}\right)F_{n\kappa}(r), \tag{36}$$

where $E_{n\kappa} \ne -M + c_s$, so that in the presence of exact spin symmetry, $c_s = 0$ and only positive energy states can exist (since the energy eigenvalues can be positive or negative). With this condition, the lower spinor can then be normalized. We therefore express the lower radial spinor as

$$G_{n\kappa}(r) = \frac{\alpha}{M+E_{n\kappa}-c_s}\left\{\begin{array}{l}\left(\frac{l_\kappa}{\alpha r}+\frac{\left(b+\frac{1}{2}\right)e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)}-\alpha\right)F_{n\kappa}(r)+\frac{\left(n+2a+2b+1\right)\Gamma\left(n+2a+1\right)}{\Gamma(n)\Gamma(2a+2+}\times\\ A_n^{a,b}\left(e^{-\alpha r}\right)^{a+1}\left(1-e^{-\alpha r}\right)^{b+\frac{1}{2}}\times{}_2F_1\left(-n+1, n+2(a+b+1), 2(a+1), e^{-\alpha r}\right)\end{array}\right\} \tag{37}$$

for $E_{n\kappa} \ne -M + c_s$

Thus, the lower spinor is expressed as

$$\phi(\vec{r}) = \frac{i\alpha}{r\left(M+E_{n\kappa}-c_s\right)}\left\{\begin{array}{l}\left(\frac{l_\kappa}{\alpha r}+\frac{\left(b+\frac{1}{2}\right)}{\left(e^{-\alpha r}-1\right)}-\alpha\right)F_{n\kappa}(r)+\left(n+2a+2b+1\right)\Gamma\left(n+2a+1\right)\times\\ A_n^{a,b}\left(e^{-\alpha r}\right)^{a+1}\left(1-e^{-\alpha r}\right)^{b+\frac{1}{2}}\times{}_2F_1\left(-n+1, n+2(a+b+1), 2(a+1), e^{-\alpha r}\right)\end{array}\right\}Y^{\tilde{l}}_{jm}\left(\theta,\varphi\right) \tag{38}$$

P-spin symmetric limit

It has been shown by [7] that there is a connection between P-spin symmetry and near equality of the time component of a vector potential and the scalar potential, $V(r) \approx -S(r)$. Also, Meng et al., showed that if $\frac{d}{dr}\left(V(r)+S(r)\right) = \frac{d\sum(r)}{dr} = 0$ or $\sum(r) = c_{ps}$, the P-spin symmetry is exact in the Dirac equation [68]. In this symmetry,

$$\Delta(r) = -\frac{V_0 e^{-\alpha r}}{1-e^{-\alpha r}}+\frac{\alpha\delta V_0 e^{-\alpha r}}{2r'_0\left(1-e^{-\alpha r}\right)^2}\left(j(j+1)-l(l+1)-\tfrac{3}{4}\right) \tag{39}$$

Therefore, if one chooses the following parameters $\tilde{l}_\kappa = \kappa + H - 1, \tilde{\beta}_{n\kappa} = E_{n\kappa} - M - c_{ps}$ and $\tilde{\gamma}^2_{n\kappa} = \left(M + E_{n\kappa}\right)\left(M - E_{n\kappa} + c_{ps}\right)$. Eq. (21) takes the form

$$\left\{\frac{d^2}{dr^2}-\frac{\tilde{l}_\kappa\left(\tilde{l}_\kappa+1\right)}{r^2}+\tilde{\beta}_{n\kappa}\left[\frac{V_0 e^{-\alpha r}}{1-e^{-\alpha r}}-\frac{\alpha\delta V_0 e^{-\alpha r}}{2r'_0\left(1-e^{-\alpha r}\right)^2}\left(j(j+1)-l(l+1)-\tfrac{3}{4}\right)\right]-\tilde{\gamma}^2_{n\kappa}\right\}G_{n\kappa}(r) = 0 \tag{40}$$

The negative energy solution of Eq. (40) in the P-spin symmetric limit $V(r) \approx -S(r)$ can be obtained directly via the symmetric solution by making the following parametric mappings.

$$F_{n\kappa}(r) \to G_{n\kappa}(r), \kappa \to \kappa - 1, V(r) \to -V(r), E_{n\kappa} \to -E_{n\kappa}, c_s \to c_{ps} \tag{41}$$

Following the above procedure, we obtain the energy spectrum for the Dirac hole states in the presence of P-spin symmetry as

$$E_{n\kappa} = -\frac{\alpha^2}{\tilde{\beta}_{n\kappa}}\left\{\tilde{l}_\kappa\left(\tilde{l}_\kappa+1\right)c_0-\left(\frac{\frac{Z\tilde{\beta}_{n\kappa}}{2\alpha}}{\left(n+\frac{1}{2}+\tilde{\delta}^{\pm}_\kappa\right)}-\frac{n+\frac{1}{2}+\tilde{\delta}^{\pm}_\kappa}{2}\right)^2\right\}-M, \tag{42}$$

with

$$\tilde{\delta}^{\pm}_\kappa = \sqrt{\left(\tilde{l}_\kappa+\tfrac{1}{2}\right)^2+\tilde{\chi}^{\pm}_{n\kappa}}$$

and
$$\tilde{\chi}^{\pm}_{n\kappa} = \begin{cases}\frac{-r_0^2 V_{LS}^{(o)} ze^2\tilde{\beta}_{n\kappa}\left(l+1\right)}{2r_0'\hbar^2}, & for\ \kappa = \left(j+\frac{1}{2}\right) > 0\\ \frac{-r_0^2 V_{LS}^{(o)} ze^2\tilde{\beta}_{n\kappa} l}{2r_0'\hbar^2}, & for\ \kappa = -\left(j+\frac{1}{2}\right) < 0\end{cases}$$

The allowed values for the adjustable parameter are now $r'_0 \ge \frac{\left(k+H-\frac{1}{2}\right)^2 r_0^2 V_{LS}^{(o)} ze^2\tilde{\beta}_{n\kappa}l_{\max}}{2\hbar^2}$ for both $\kappa < 0$ and $\kappa > 0$

The lower radial spinor for the Dirac hole wave functions can then be calculated to be

$$G_{n\kappa}(r) = \tilde{N}_n\left(e^{-\alpha r}\right)^{\tilde{a}}\left(1-e^{-\alpha r}\right)^{\tilde{b}+\frac{1}{2}}P_n^{\left(2\tilde{a},2\tilde{b}\right)}\left(1-2e^{-\alpha r}\right) \tag{43}$$

with

$$\tilde{a} = \sqrt{\tilde{\eta}_\kappa+\tilde{\sigma}^2}$$
$$\tilde{b} = \sqrt{\tfrac{1}{4}+\tilde{\Lambda}_\kappa+\tilde{\chi}_{n\kappa}}$$

such that

$\tilde{\eta}_\kappa = \tilde{l}_\kappa\left(\tilde{l}_\kappa+1\right)c_0, \tilde{\Lambda}_\kappa = \tilde{l}_\kappa\left(\tilde{l}_\kappa+1\right), \tilde{\sigma} = \frac{\tilde{y}_{n\kappa}}{\alpha}$, and $\tilde{N}_n$ is the new normalization constant.

Furthermore, the upper- spinor component of the Dirac hole wave functions can be calculated from

$$F_{n\kappa}(r) = \frac{\alpha}{M+E_{n\kappa}-c_{ps}}\left(\frac{d}{ds}-\frac{\left(\kappa+H\right)}{r}\right)G_{n\kappa}(r) \tag{44}$$

where $M + E_{n\kappa} - c_{ps}$ and only negative energy states exist in the presence of exact P-spin symmetry $\left(c_{ps} = 0\right)$. Hence, on substituting Eq. (43) into Eq. (44), one obtains

$$F_{n\kappa}(r) = \frac{\alpha}{M+E_{n\kappa}-c_{ps}}\left\{\begin{array}{l}\left(\frac{\left(\tilde{b}+\frac{1}{2}\right)}{\left(e^{\alpha r}-1\right)}-\frac{\tilde{l}_\kappa}{\alpha r}-\tilde{a}\right)G_{n\kappa}(r)+\frac{\left(n+2\tilde{a}+2\tilde{b}+1\right)\Gamma\left(n+2\tilde{a}+1\right)\beta_n^{\tilde{a},\tilde{b}}}{\Gamma(n)\Gamma\left(2\tilde{a}+2\right)}\\ \times\left(e^{-\alpha r}\right)^{\tilde{a}+1}\left(1-e^{-\alpha r}\right)^{\tilde{b}+\frac{1}{2}}\ {}_2F_1\left(-n+1, n+2\left(\tilde{a}+\tilde{b}+1\right), \left(\tilde{a}+1\right), e^{-\alpha r}\right)\end{array}\right\} \tag{45}$$

for $E_{n\kappa} \ne M + c_{ps}$,

where

$$\beta_n^{\tilde{a},\tilde{b}} = \sqrt{\frac{\left(n+\tilde{a}+\tilde{b}+\frac{1}{2}\right)\Gamma\left(n+2\tilde{a}+1\right)\Gamma\left(n+2\tilde{a}+2\tilde{b}+1\right)}{n!\left(n+\tilde{b}+\frac{1}{2}\right)\Gamma\left(n+2\tilde{b}+1\right)\Gamma\left(2\tilde{a}\right)\Gamma\left(2\tilde{a}+1\right)}} \quad (46)$$

Hence, the upper and lower Dirac holespinors are given by

$$\xi(\vec{r}) = \frac{\alpha}{r\left(M+E_{n\kappa}-c_{ps}\right)}\left\{\begin{array}{l}\left(\frac{\tilde{b}+\frac{1}{2}}{e^{\alpha r}-1}-\frac{\tilde{l}_\kappa}{\alpha r}-\tilde{a}\right)G_{n\kappa}(r)+\frac{\left(n+2\tilde{a}+2\tilde{b}+1\right)\Gamma\left(n+2\tilde{a}+1\right)\beta_n^{\tilde{a},\tilde{b}}}{\Gamma(n)\Gamma\left(2\tilde{a}+2\right)} \\ \times\left(e^{-\alpha r}\right)^{\tilde{a}+1}\left(1-e^{-\alpha r}\right)^{\tilde{b}+\frac{1}{2}}{}_2F_1\left(-n+1,n+2\left(\tilde{a}+\tilde{b}+1\right),\left(\tilde{a}+1\right),e^{-\alpha r}\right)\end{array}\right\}$$

$$\times\frac{1}{\sqrt{2l+1}}\begin{pmatrix}\pm\sqrt{l\pm m+\frac{1}{2}}\gamma_{l,m-\frac{1}{2}}(\theta,\varphi) \\ \pm\sqrt{l\mp m+\frac{1}{2}}\gamma_{l,m+\frac{1}{2}}(\theta,\varphi)\end{pmatrix}, for\, j = l\pm\frac{1}{2},(j\geq 2) \quad (46)$$

explicitly and

$$\psi(\vec{r}) = \frac{i\Gamma\left(n+2\tilde{a}+2\tilde{b}+1\right)}{n!\Gamma\left(2\tilde{a}+2\right)}\beta_n^{\tilde{a},\tilde{b}}r^{-1}\left(e^{-\alpha r}\right)^{\tilde{a}}\left(1-e^{-\alpha r}\right)^{\tilde{b}+\frac{1}{2}}$$

$$\times{}_2F_1\left(-n+2\tilde{a}+2\tilde{b}+1,2\tilde{a}+1,e^{-\alpha r}\right)\times\gamma_{l,m-\frac{1}{2}}^{\tilde{l}}\left(\theta,\varphi\right) \quad (47)$$

respectively with $\tilde{l} = l+1$

Numerical Computation

The numerical computation for the proposed potential is suitable for the description of both particle (negative values) and anti-particles (positive values) which is shown from Tables 1-4. The numerical values for these potential on Dirac equation decreases with an increase in the screening parameter with fixed pseudo coupling constant. The numerical values also decreases with a decrease in quantum state. The findings of the research work is in total agreement with that of existing literature which authenticate the accuracy of both theoretical and numerical solutions.

Conclusion

In this paper, we have investigated Dirac equation within the framework of spin and pseudospin symmetries for a spin-orbit coupled Hulthen potential including Coulomb-like tensor interaction. We have also obtained the energy levels and corresponding Dirac spinors(lower and upper wave functions) of this potential using the NU method with the new approximation scheme introduced by Dong [63]. Our results would be of great significant in the study of meson-meson interaction and would find

Table 1: The energy of pseudospin symmetry for a spin-orbit coupled potential including a Coulomb-like tensor interaction for $m = 1.0Fm^{-1}$, $\delta = 0.1Fm^{-1}$, $C_{ps} = -1.0$, $\beta = -1.0$

$\tilde{l}$	n,k	$E_{n,k}^{ps}\left(H=V=0\right)$ $\alpha = 0.1$	$E_{n,k}^{ps}\left(H=V=0\right)$ $\alpha = 0.2$	$E_{n,k}^{ps}\left(H=V=0\right)$ $\alpha = 0.3$	$E_{n\,k}^{ps}\left(H=V=\ \right)$ 0.4
1	1,-1	0.8459437500	0.776743750	0.694743750	0.5999437500
2	1,-2	0.8259437500	0.696743750	0.514743750	0.279943750
3	1,-3	0.7859437500	0.536743750	0.154743750	-0.360056250
4	1,-4	0.7259437500	0.296743750	-0.385256250	-1.320056250
1	0,2	0.1946555556	-0.038044444	-0.392544444	-.8688444444
2	0,3	0.1346555556	-0.278044444	-0.932544444	-1.828844444
3	0,4	0.0546555555	-0.598044444	-1.652544444	-3.108844444
4	0,5	-0.0453444444	-0.998044444	-2.552544444	-4.708844444
5	0,6	-0.1653444444	-1.478044444	-3.632544444	-6.628844444
6	0,7	-0.3053444444	-2.038044444	-4.892544444	-8.868844444
7	0,8	-0.4653444444	-2.678044444	-6.332544444	-11.42884444
8	0,9	-0.6453444444	-3.398044444	-7.952544444	-14.30884444
9	0,10	-0.8453444444	-4.198044444	-9.752544444	-17.50884444
10	0,11	-1.0653444440	-5.078044444	-11.73254444	-21.02884444
11	0,12	-1.3053444440	-6.038044444	-13.89254444	-24.86884444

Table 2: The energy of pseudospin symmetry for a spin-orbit coupled potential including a Coulomb-like tensor interaction for $m = 1.0Fm^{-1}$, $\delta = 0.1Fm^{-1}$, $C_{ps} = -2.0$, $\beta = -1.0$

$\tilde{l}$	n,k	$E^{ps}_{n,k}(H = V = 0)$ $\alpha = 0.1$	$E^{ps}_{n,k}(H = V = 0)$ $\alpha = 0.2$	$E^{ps}_{n,k}(H = V = 0)$ $\alpha = 0.3$	$E^{ps}_{n,k}(H = V = 0)$ $\alpha = 0.4$
1	1,-1	0.8459437500	0.7767437500	0.694743750	0.599943750
2	1,-2	0.805943750	0.6167437500	0.334743750	-0.0400562500
3	1,-3	0.725943750	0.2967437500	-0.3852562500	-1.320056250
4	1,-4	0.605943750	-0.183256250	-1.465256250	-3.240056250
1	0,2	0.134655555	-0.278044444	-0.932544444	-1.828844444
2	0,3	0.014655555	-0.758044444	-2.012544444	-3.748844444
3	0,4	-0.145344444	-1.398044444	-3.452544444	-6.308844444
4	0,5	-0.345344444	-2.198044444	-5.252544444	-9.508844444
5	0,6	-0.585344444	-3.158044444	-7.412544444	-13.34884444
6	0,7	-0.865344444	-4.278044444	-9.932544444	-17.82884444
7	0,8	-1.185344444	-5.558044444	-12.81254444	-22.94884444
8	0,9	-1.545344444	-6.998044444	-16.05254444	-28.70884444
9	0,10	-1.945344444	-8.598044444	-19.65254444	-35.10884444
10	0,11	-2.385344444	-10.35804444	-23.61254444	-42.14884444
11	0,12	-2.865344444	-12.27804444	-27.93254444	-49.82884444

Table 3: The energy of pseudospin symmetry for a spin-orbit coupled potential including a Coulomb-like tensor interaction for $m = 1.0Fm^{-1}$, $\delta = 0.1Fm^{-1}$, $C_{ps} = -1.0$, $\beta = -1.0$

$\tilde{l}$	n,k	$E^{ps}_{n,k}(H = 0.3, V = 0)$ $\alpha = 0.1$	$E^{ps}_{n,k}(H = 0.5, V = 0)$ $\alpha = 0.1$	$E^{ps}_{n,k}(H = 0.7, V = 0)$ $\alpha = 0.1$	$E^{ps}_{n,k}(H = 1.0, V = 0)$ $\alpha = 0.1$
1	1,-1	0.8480437500	0.8484437500	0.8480437500	0.8480437500
2	1,-2	0.8340437500	0.8384437500	0.8420437500	0.8420437500
3	1,-3	0.8000437500	0.8084437500	0.8160437500	0.8160437500
4	1,-4	0.7460437500	0.7584437500	0.7700437500	0.7700437500
1	0,2	0.1787555556	0.1671555556	0.1547555556	0.1547555556
2	0,3	0.1127555556	0.0971555555	0.080755555	0.0807555555
3	0,4	0.0.267555555	0.0071555555	-0.0132444444	-0.0132444444
4	0,5	-0.0792444444	-0.1028444444	-0.1272444444	-0.1272444444
5	0,6	-0.2052444444	-0.2328444444	-0.2612444444	-0.2612444444
6	0,7	-0.3512444444	-0.3828444444	-0.4152444444	-0.4152444444
7	0,8	-0.5172444444	-0.5528444444	-0.5892444444	-0.5892444444
8	0,9	-0.7032444444	-07428444444	-0.7832444444	-0.7832444444
9	0,10	-0.9092444444	-0.9528444444	-0.9972444444	-0.9972444444
10	0,11	-1.135244444	-1.1828444444	-1.2312444444	-1.2312444444
11	0,12	-1.381244444	-1.4328444444	-1.4852444444	-1.4852444444

Table 4: The energy of pseudospin symmetry for a spin-orbit coupled potential including a Coulomb-like tensor interaction for $m = 1.0Fm^{-1}$, $\delta = 0.1Fm^{-1}$, $C_{ps} = -2.0$, $\beta = -1.0$

$\tilde{l}$	n,k	$E_{n,k}^{ps}(H=0.3, V=0)$ $\alpha = 0.1$	$E_{n,k}^{ps}(H=0.5, V=0)$ $\alpha = 0.1$	$E_{n,k}^{ps}(H=0.7, V=0)$ $\alpha = 0.1$	$E_{n,k}^{ps}(H=1.0, V=0)$ $\alpha = 0.1$
1	1,-1	0.850143750	0.850943700	0.850143750	0.850143750
2	1,-2	0.822143750	0.830943750	0.838143750	0.838143750
3	1,-3	0.754143750	0.770943750	0.786143500	0.786143750
4	1,-4	0.646143750	0.670943750	0.694143700	0.694143700
1	0,2	0.102855555	0.079655555	0.548555555	0.054855555
2	0,3	-0.029144444	-0.060344444	-0.093144444	-0.093144444
3	0,4	-0.201144444	-0.240344444	-0.281144444	-0.281144444
4	0,5	-0.413144444	-0.460344444	-0.509144444	-0.509144444
5	0,6	-0.665144444	-0.720344444	-0.777144444	-0.777144444
6	0,7	-0.957144444	-1.020344444	-1.085144444	-1.085144444
7	0,8	-1.289144444	-1.360344444	-1.433144444	-1.433144444
8	0,9	-1.661144444	-1.740344444	-1.821144444	-1.821144444
9	0,10	-2.073144444	-2.160344444	-2.249144444	-2.249144444
10	0,11	-2.525144444	-2.620344444	-2.717144444	-2.717144444
11	0,12	-3.017144444	-3.120344444	-3.225144444	-3.225144444

References

1. Ginocchio JN. Phys. Rev. 2005; 414(4-5):165.
2. Page PR, Goldman T, Ginocchio JN. Phys. Rev. Lett. 2001; 86:204.
3. Hassanabadi H, Yazarloo BH, Salehi N. Ind. J. Phys. 2014; 88(4):405.
4. Bohr A, Hamamoto I, Mottelson BR. Phys. Scr. 1982; 26:267.
5. Dudek J, Nazarewicz W, Szymanski Z, Leander GA. Phys. Rev. Lett. 1987; 59:1405.
6. Ginocchio JN, Leviatan A, Meng J, Zhou SG. Phys. Rev. C. 2004; 69:03430.
7. Ginocchio JN. Rev. Lett. 1997; 78(3):436.
8. Hecht KT, Adler A. Nucl. Phys. A. 1969; 137:129.
9. Arima A, Harvey M, Shimizu K. Phys. Lett. B. 1969; 30:517.
10. Ikhdair SM, Sever R. Appl. Math.Comput. 2010; 216:911.
11. Ikhdair SM, Sever R. Appl. Math. Com. 2010; 216:911.
12. Lisboa R, Malheiro M, de Castro AS, Alberto P, Fidhais M. Phys. Rev. C. 2004; 69:024319.
13. Moshinsky M, Szczepanika A. J. Phys. A. Math. Gen. 1989; 22:L 817.
14. Chen TS, Lu HF, Meng J, Zhang SQ, Zhou SG. Chin. Phys. Lett. 2003; 20:358.
15. Berkdemir C. Nucl. Phys. A. 2009; 821:262-263.
16. Guo JY, Sheng ZQ. Phys Lett. A. 2005; 338:90-96.
17. Xu Q, Zhu SJ. Nucl. Phys. A. 2006; 768:161-169.
18. Zhang LH, Li XP, Jia CS. Phys. Lett. A. 2008; 372:2201.
19. Soylu A, Bayrak O, Boztosun I. J. Phys. A: Math. Theor. 2008; 41:065308.
20. Jia CS, Guo P, Pong XL. J. Phys. A: Math Gen. 2006; 39:7737.
21. Jia CS, Guo P, Diao YF, Yi LZ, Xie XJ. Eur. Phys. J. A. 2007; 34:41.
22. Xu Y, He S, Jia CS. J. Phys. A: Math. Theor. 2008; 41:255302.
23. Jia CS, Chen T, Ciu LG. Phys. Lett. A. 2009; 373:1621 .
24. Taskin F. Int. J. Theo. Phys. 2009; 48:1142.
25. Kukulin VI, Loyla G, Moshinsky M. Phys. Lett. A. 1991; 158:19.
26. Alberto P, Lisboa R, Malheiro M, de Castro AS. Phys. Rev. C. 2005; 71:034313.
27. Aydogdu O, Server R. Few - Body Syst. 2010; 47:193.
28. Sever R, Tezcan C, Aktas M, Yesilitas O. J. Math. Chem. 2007; 43:845.
29. Ikhdair SM, Sever R. J. Mol. Struct. Theochem. 2007; 806:155.
30. Dong SH, Gu XY, Ma ZQ, Dong S. Int. J. Mod. Phys. E. 2002; 11:483.
31. Mckeon DGC, Leeuwen GV. Mod. Phys. Lett. A. 2002; 17:1961.
32. Ikhadair SM, Sever R. Cent. Eur. J. Phys. 2007; 5:516.
33. Hamzavi M, Hassanabadi H, Rajabi AA. Int. J. Mod. Phys. E. 2010; 19:2189.
34. Hamzavi M, Hassanabadi A, Rajabi AA. Phys. Lett. A. 2010; 25:2497.
35. Hamzavi M, Rajabi AA, Hassanbadi H. Phys. Lett. A. 2010; 374:4303.
36. Zhang LH, Li XP, Jia CS. Phys. Scr. 2009; 80:035003.
37. Dong SH, Lozada-Cassou M. Int. J. Mod. Phys. E. 2004; 13:917.
38. Ikhdair SM, Server R. Int. J. Mod. Phys. C. 2008; 19:1425.
39. Ikhdair SM. Mol. Phys. 2012; 110:1415.
40. Ikhadair SM, Server R. J. Phys. A. Math. Theor. 2011; 44:345301-29.
41. Dong SH. Appl. Math. Lett. 2003; 16:199.
42. Zhang MC, Huang-Fu GQ, An B. Phys. Scr. 2009; 80:065018.
43. Wei GF, Dong SH. Phys. Lett. B. 2010; 686:288.

many applications in nuclear, chemical and molecular physics especially in the recently reported result of neutron-proton pairs in heavy nuclei using perturbation theory [69].

44. Ikhdair SM. Phys. Ser. 2011; 83:015010.
45. Oyewunmi KJ, Akinpelu FO, Agboola AD. Int. J. Theor. Phys. 2017; 47:1039.
46. Pahlavani MR, Alavi SA. Commun. Theor. Phys. 2012; 58:739.
47. Ikhadair SM. Cent. Eur. J. Phys. 2011; 101361.
48. Ikot AN, Akpabio LE, Uwah EJ. Elect J. Theor Phys. 2011; 8(25):225.
49. Berkdemir C, Chen YF. Phys. Scr. 2009; 79:035003.
50. Nikiforov AF, Uvarov VB. Special Function of Mathematical Physics. (Basel, Birkhauser); 1998.
51. Bag M, Panja MM, Dutt R, Varshni YP. Phys. Rev. A. 1992; 46:6059.
52. Ma ZQ, Xu W. Europhys. Lett. 2005; 69:685.
53. Ciffuld, Hall RL, Saad N. J. Phys. A. 2003; 36:11807.
54. Cooper F, Sukhatme UP. Super Symmetry in Quantum Mechanics. (Singapore World Scientific); 2001.
55. Ikhadair SM, Lever R. J. Math. Chem, 2007; 42:461.
56. Bayrak O, Kocak G, Boztosun I. J. Phys. A Math Gen. 2006; 39:11521.
57. Hulthen L. Ark Mat. Astron Fys. A. 1942; 28:5.
58. Jia CS, Li Y, Sun Y, Liu JY, Sun LT. Phys. Lett. A. 2003; 311:115.
59. Jia CS, Zeng XL, Sun LT. Phys Lett. A. 2002; 294:185.
60. Ikhadair SM. arXiV: 0901. 0801VI [quant-ph]; 2009.
61. Tezcan C, Sever R. Int. J. Theor. Phys. 2009; 46:337.
62. Greiner W, Muller B. Quantum mechanics symmetries. 2nd Ed. Berling: Springer; 1994.
63. Dong SH. Commun. Theor. Phys. 2011; 55:969-971.
64. Jeffreys H, Jeffereys BS. Methods of mathematical physics, 3rd Ed. Cambridge, England: Cambridge University press; 1988.
65. Abramowitz M, Stegun IA. Handbook of the Mathematical functions with formulas, Graphs and Mathematical Tables and Printing. (New York Dover); 1972.
66. Antia AD, Essien IE, Umoren EB, Eze CC. Adv. Phys. Theor. Appl. 2015; 44:1-13.
67. Ahmadov HI, Jafarzade ShI, Qocayeva MV. arXir: 1606.08035VI [math-ph]; 2016.
68. Akcay H. Phys. Lett. A. 2009; 373:616.
69. Khanna KM, Kanyeki GF, Rotic SK, Torongey PK, Ameka SE. Indian J. Pure and Applied Phys. 2010; 48:7.