The Significance of Limit Cycles in a Neural Model with Poisson and Gauss Connectivity

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Abstract
Isolated randomly interconnected nets with chemical markers and noise are investigated, which follow Poisson or Gauss distribution. The obtained results reveal limit cycles. The Poisson limit cycles are primarily large and complex, while the Gauss ones are regularly small. The Poisson limit cycles have various types depending on the shape and time of the transient part, whereas the Gauss ones have the same form, plain, with no particular types of the transient component and are small. The epileptic MEGs follow Poisson distributions with high magnetic potentials and noise are investigated, which follow Poisson or Gauss distributions. The obtained results reveal limit cycles. The Poisson limit cycles have various types depending on the shape and time of the transient part, whereas the Gauss ones have the same form, plain, with no particular types of the transient component and are small. The epileptic MEGs follow Poisson distributions with high magnetic potentials and noise are investigated, which follow Poisson or Gauss distributions. The obtained results reveal limit cycles. The Poisson limit cycles have various types depending on the shape and time of the transient part, whereas the Gauss ones have the same form, plain, with no particular types of the transient component and are small.

Keywords: Network models, Poisson distribution, Gauss distribution, Limit cycles.

The Neural Net Model
The basic hypotheses of this model have been described in detail previously [1-4]. In short, a neural net with N neurotransmitters (markers) is supposed to be constructed of A neurons. A portion h (0<h<1) of them are inhibitory while the rest are excitatory. Each neuron receives on average, μ EPSPs (Excitatory Post-Synaptic Potentials) and μ IPSPs (Inhibitory Post-Synaptic Potentials). Κ (K) is defined the volume of the PSP produced by an excitatory (inhibitory) component. The neurons are also characterized by the absolute refractory period (r) and the synaptic delay r (r<r<2r). For our theory, r was given the value r=1 when refractoriness was assumed, and r≠0 if not. If a neuron fires concurrently at time t, then all neural activity resulting from this primary activity will be limited to times t+r, t+2r, .... If a neuron fires at time t, it produces the PSPs after a synaptic delay r. PSPs arriving at a neuron are summed at once, and if this sum is greater or equal to θ, then the neuron will fire, or else it will be inactive. If the PSPs are below the threshold (θ) then they will stay with or without decrement for a period called the summation time. The firing is temporary and causes the neuron to be insensitive to additional stimulation for the time of a refractory period [5-10].

Poisson distribution
Following the suppositions of previous papers [1-10], the expectation value of the neural activity <a_n> at t=(n+1) r, (i.e. the average value) of a_n generated by a collection of netlets with the same parameters (κ, μ, h, K, A, θ) at t=n r with 2 markers m and n, is given by:

\[
<q_{an}>= \left(1-a_n\right) \left[m_a \sum_{i=0}^{m_a-1} \sum_{l=0}^{l_a} P(Q_i T_j \theta_1) + (1-m_a) \sum_{l=0}^{l_a} P(Q_i T_j \theta_2)\right]
\]

(1)

where \(P_i, Q_i, P', Q'\) are the probabilities that the neuron will receive j EPSPs, j IPSPs or j’-EPSPs, j’-IPSPs, at time t=(n+1)r in the subsystems a or b. These probabilities are given by:

\[
P_i = \exp \left(-a_{n_a} \mu_a (1-h_a) m_a \right) \left(-a_{n_a} \mu_a h_a m_a \right) / i!
\]

\[
Q_i = \exp \left(-a_{n_a} \mu_a (1-h_a) m_a \right) \left(-a_{n_a} \mu_a h_a m_a \right) / i!
\]

\[
P'_i = \exp \left(-a_{n_a} \mu_a (1-h_a) (1-m_a) \right) \left(-a_{n_a} \mu_a (1-h_a) (1-m_a) \right) / i'!
\]

\[
Q'_i = \exp \left(-a_{n_a} \mu_a h_a (1-m_a) \right) \left(-a_{n_a} \mu_a h_a (1-m_a) \right) / i'!
\]

(2)

The higher limits in the sums in equation (1) are given by:

\[
l_{max} = A \alpha_a \mu_a (1-h_a) m_a
\]

\[
l_{max} = A \alpha_a \mu_a (1-h_a) (1-m_a)
\]

\[
i_{max} = A \alpha_a h_a m_a
\]

\[
i_{max} = A \alpha_a h_a (1-m_a)
\]

(3)

\[
T_a(\theta) \text{ and } T_{ab}(\theta)
\]

are defined as the probabilities that the
instantaneous neural thresholds are equal to or less than \( \theta_a \) and \( \theta_b \) in subsystems a and b and are given by:

\[
T_{\theta_a}(\theta_a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left( -\frac{x^2}{2} \right) dx \quad T_{\theta_b}(\theta_b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left( -\frac{x^2}{2} \right) dx
\]  

In the general case with \( N \) chemical markers \( m_j \) \( (j=1,\ldots,N) \) eq. (1) takes the form:

\[
\angle \alpha_{n+1} = (1 - \alpha_n) \sum_{j=1}^{N} m_j \sum_{i_j=0}^{l_{max,j}} \sum_{i_{j'}=0}^{l_{max,j'}} m_{i_j} Q_{i_{j'}} T_{\delta_{i_j}}(\theta_i) + (1 - \sum_{j=1}^{N} m_j) \sum_{i_j=0}^{l_{max,j}} \sum_{i_{j'}=0}^{l_{max,j'}} \alpha_{n+1} Q_{i_{j'}} T_{\delta_{i_j}}(\theta_i)
\]

**Gauss distribution**

If the numbers \( l_a, l_b, i_a, \) and \( i_b \) are sufficiently large, their distributions may be approximated by Gaussian distributions about their average values.

Consequently, the probability that a neuron with marker \( a \) or \( b \) will obtain a definite number of EPSPs or IPSPs that will move the membrane potential closer to or further away from the instantaneous threshold will be given by:

\[
P_a(\alpha_{n+1}, m_a, \overline{\theta_a}) = \frac{1}{\sqrt{2\pi}} \int_{x_{a,n+1}}^{\infty} \exp\left( -\frac{x^2}{2} \right) dx \quad \alpha_{n+1} = (\overline{\theta_a} - \overline{\theta_{a+1}}) / \overline{\delta_{a,n+1}}
\]

and

\[
P_b(\alpha_{n+1}, m_b, \overline{\theta_b}) = \frac{1}{\sqrt{2\pi}} \int_{x_{b,n+1}}^{\infty} \exp\left( -\frac{x^2}{2} \right) dx \quad \alpha_{n+1} = (\overline{\theta_b} - \overline{\theta_{b+1}}) / \overline{\delta_{b,n+1}}
\]

In consequence, the possibilities \( T_{\theta_a}(\theta_a) \) and \( T_{\theta_b}(\theta_b) \) that the instantaneous threshold of a neuron in subsystems a and b is equal to or less than \( \theta_a \) or \( \theta_b \) will be given by:

\[
T_{\theta_a}(\theta_a) = \frac{1}{\sqrt{2\pi}} \int_{(\overline{\theta_a} - \overline{\theta_{a+1}})}^{\infty} \exp\left( -\frac{x^2}{2} \right) dx
\]

Accordingly, the firing probabilities \( P(\alpha_{n+1}, \delta_{a+b}, \delta_{a-b}) \) and \( P'(\alpha_{n+1}, \delta_{a+b}, \delta_{a-b}) \) that a neuron in subpopulations a and b, will receive PSPs exceeding the threshold at time \( t=(n+1)\tau \) will be given by:

\[
P(\alpha_{n+1}, \delta_{a+b}) = P(\alpha_{n+1}, m_a, \overline{\theta_a}) \sum_{j=0}^{l_a} \sum_{i=0}^{l_{max,j}} T_{\theta_j}(\theta_j) = P(\alpha_{n+1}, m_a, \overline{\theta_a}) \sum_{j=0}^{l_a} \sum_{i=0}^{l_{max,j}} T_{\theta_j}(1K^* + iK^*)
\]

and

\[
P'(\alpha_{n+1}, \delta_{a+b}) = P(\alpha_{n+1}, m_b, \overline{\theta_b}) \sum_{j=0}^{l_b} \sum_{i=0}^{l_{max,j}} T_{\theta_j}(\theta_j) = P(\alpha_{n+1}, m_b, \overline{\theta_b}) \sum_{j=0}^{l_b} \sum_{i=0}^{l_{max,j}} T_{\theta_j}(1K^* + iK^*)
\]

The general case for an isolated noisy net with \( N \) markers \( m_1, m_2, \ldots, m_N \) where \( m_i \) is the fraction of neurons with the \( i \)th marker at time \( t=(n+1)\tau \) is given by:

\[
\angle \alpha_{n+1} = (1 - \alpha_n) \sum_{j=1}^{N} m_j P_j(\alpha_{n+1}, m_j, \overline{\theta_j}) T_{\theta_j}(1K^* + iK^*)
\]

**Magnetic field**

The description of the magnetic field has been described in detail in our previous work [11,12]. In the general case, where the neural net has \( N \) chemical markers, it is given by the following equation:

\[
B_n = \frac{1}{2} \mu_0 \varepsilon_0 \alpha_{n+1} \sum_{j=1}^{N} m_j \mu_j (1-h_j) m_j \sum_{j=1}^{N} \mu_j h_j m_j = \frac{1}{2} \mu_0 \varepsilon_0 \alpha_{n+1} \sum_{j=1}^{N} \mu_j (1-h_j) m_j \sum_{j=1}^{N} \mu_j h_j m_j
\]
the time is quantized with the unit quantum time the synaptic delay $\tau$, it is not essential to differentiate the neural activity $a_n$ as it is usually done for neural models of continuum time, but instead it is taken the difference $\Delta a_n = a_{n+1} - a_n$ which is included as a term in the description of the magnetic field $B_n$.

**Results**

Evaluated the obtained phase diagrams we observed the following:

The Poisson phase diagrams are complicated and have closed and open hysteresis loops while the Gauss ones are simple and have only open hysteresis loops (Figure 1A,B).

Comparing the obtained limit cycles we observed the following:

The Poisson limited cycles are mostly large and complex, while the Gauss ones are generally small. The transient part in Poisson limited cycles is complex while in Gauss is regularly plain (Figure 2A,B). The Poisson limited cycles have various forms depending on the structure and time of the transient part, while the Gauss ones have the same shape, plain, without particular types of the transient component and are small.

**Discussion**

The Poisson distribution expresses the probability of a given number of events occurring in a fixed time interval of space, distance, area or volume, with the assumption that these events occur with a known average rate and independently of the time since the last event. Some applications that follow Poisson distribution are: birth defects, genetic mutations, rare diseases, car accidents, traffic flow. If the number of events is very large, then the Gaussian or normal distribution may be used to describe physical events. The Gaussian distribution is a probability distribution that associates the normal random variable with a cumulative probability. It is an arrangement of a data set in which most values cluster in the middle of the range and the rest taper off symmetrically toward either extreme. The Gaussian distribution is a very common continuous probability distribution and is often used in the natural and social sciences to represent real-valued random variables whose distributions are not known. It is important because lots of variables studied in education and psychology are normally distributed, like reading ability, job satisfaction and memory.

In recent years the consequence of structure on function and dynamic behavior in neural nets has been also a topic of considerable attention because the main idea is that this connectivity is given by a binomial distribution. Probabilistic
neural nets were investigated using Poisson or Gauss distributions of inter-neuronal connectivity with the significant conclusion that when a neuron was connected to a relatively small number of units, a Poisson distribution law was proper but if it was connected to a great number of units then a Gauss law was a quite a good estimation. Consequently, Poisson neuronal nets may be viewed as approximately Gauss when the number of synaptic connections is relative large [13-17].

In Figure 1A,B we observe the different hysteresis curves for noisy neural nets with chemical markers with Poisson and Gauss connectivity. As we observe, a small change of ω which characterizes the spontaneous activity may lead to permanent changes in the steady-state activity of the net. Therefore, the effects of the inherent noise of isolated neural nets are functionally comparable to the effects of sustained inputs to noiseless nets [18]. Another characteristic of the hysteresis loops in both Poisson and Gauss noisy neural nets is that in the Poisson case we have open and closed phase diagrams, while in Gauss one we have only open ones. In the case of open phase diagrams, the high state activity is maintained, even with reduced inherent noise of the system, except if we introduce inhibitory inputs. Alternatively, in the case of closed ones, the high stable steady-state activity might return to the lower stable state activity either by lowering δω by introducing inhibitory inputs.

In our former work we compared the hypothetical results with the investigational findings using magnetoencephalographic (MEG) measurements in epileptic patients and healthy volunteers [11,12]. The epileptic MEGs have revealed to follow Poisson distributions with high magnetic amplitudes varying with time and repeatable at time intervals with similar characteristics like the limit cycles. Alternatively the MEGs from normal subjects had Gauss distributions. The epileptic patients had high MEG amplitudes characterized with θ (4-7Hz) or δ (2-3Hz) rhythms and absence of α-rhythm (8-13Hz) whereas the MEG from normal subjects had low amplitudes, higher frequencies and presence of α-rhythm (8-13Hz). The application of transcranial magnetic stimulation (TMS) to epileptic patients changes the distribution of the MEG from Poisson to Gauss. This was in accordance with the connectivity of the theoretical neural model [12,19,20].

Poison and Gauss distributions have been studied in other models elsewhere. Salinas [21] obtained a mathematical model for solute dynamics assuming that pores follow a Poisson distribution in the lipid phase and that their permeability’s follow a Gaussian distribution. He studied a new proposed theory, and suggested a cut-point method in order to achieve a more accurate estimation of the latency cycles. Alternatively the MEGs from normal subjects had low amplitudes, higher frequencies and absence of α-rhythm (8-13Hz) whereas the MEG from patients had high amplitudes characterized with θ (4-7Hz) or δ (2-3Hz) rhythms and absence of α-rhythm (8-13Hz). The application of transcranial magnetic stimulation (TMS) to epileptic patients changes the distribution of the MEG from Poisson to Gauss. This was in accordance with the connectivity of the theoretical neural model [12,19,20].

In conclusion, due to the fact that the Gauss distribution is a random process, the time course of this system does not exhibit limit cycles or if it does they must be very small, which is in a complete concurrence with the hypothetical neural model. The above mentioned differences are due to the fact that in Poisson connectivity the action of the system is arranged and synchronized like in epileptic discharges and consequently it would achieve limit cycles, while in Gauss connectivity the system is random and disordered as in healthy subjects and the limit cycles are especially small or they don’t exist.

Appendix

The subscript i is a marker label and indicates the properties of a subpopulation in the netlet characterized by the i-th marker.

**Parameters**

- $\mu_i$: The average number of neurons receiving excitatory postsynaptic potentials (EPSPs) from one excitatory neuron
- $\mu$: The average number of neurons receiving inhibitory postsynaptic potentials (IPSPs) from one inhibitory neuron
- $K_i^+, K_i^-$: The size of PSP produced by an excitatory/inhibitory neuron
- $\theta$: Firing thresholds of neurons
- $\tau$: Synaptic delay
- $A$: Total number of neurons
- $h_i$: Fraction of inhibitory neurons
- $m_i$: Fractions of neurons carrying the i-th marker
- $n$: The fractional number of active neurons at time $t=\pi \tau$
- $\delta$: Standard deviation of the Gaussian distribution of the neural firing thresholds.

**References**
